VIJAY SHEKHAR ACADEMY

Coaching. Excelling. Leading.

Max. Marks : 80 | **Applied Mathematics - I** | Date :

Duration : 3 Hours | **December - 2012** | Test No.

Instructions : (i) Question No. 1 is compulsory.

(ii) Attempt any three questions from the remaining five.

(iii) Figures to the right indicate full marks.

1. (a) Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cos h^2 x}}} = \cos h^2 x$. **(3)**

Solution :

$$\text{LHS} = \frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cos h^2 x}}} \quad \text{..... By using} \begin{Bmatrix} \cos h^2 x - \sin h^2 x = 1 \\ 1-\tan h^2 x = \sec h^2 x \\ \cot h^2 x - 1 = \text{cosec}\, h^2 x \end{Bmatrix}$$

$$= \frac{1}{1-\dfrac{1}{1-\dfrac{1}{-\sin h^2 x}}} = \frac{1}{1-\dfrac{1}{1+\text{cosec}\, h^2 x}} = \frac{1}{1-\dfrac{1}{\cot h^2 x}}$$

$$= \frac{1}{1-\tan h^2 x} = \frac{1}{\sec h^2 x} = \cos h^2 x = \text{RHS}$$

(b) If $u = \log[\tan x + \tan y]$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} = 2$. **(3)**

Solution :

$u = \log(\tan x + \tan y)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y}\sec^2 x$$

$$= \frac{\sec^2 x}{\tan x + \tan y}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y}$$

$\therefore \quad \text{L.H.S.} = \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y}$

$$= \frac{1}{\tan x + \tan y}[\sin 2x \sec^2 x + \sin 2y \sec^2 y]$$

$$= \frac{1}{\tan x + \tan y}\left[\frac{2\sin x \cos x}{\cos^2 x} + \frac{2\sin y \cos y}{\cos^2 y}\right] = \frac{1}{\tan x + \tan y} \cdot 2(\tan x + \tan y)$$

$$= 2 = \text{R.H.S.}$$

(c) If $u = \frac{x+y}{1-xy}, v = \tan^{-1} x + \tan^{-1} y$, find $\frac{\partial(u, v)}{\partial(x, y)}$. **(3)**

Solution :

We have, $\frac{\partial u}{\partial x} = \frac{(1-xy)\cdot 1-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)1-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

(d) Show that $\log[1+\sin x] = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots\ldots\ldots$ **(3)**

Solution :

$$\log(1+\sin x) = \sin x - \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3} - \frac{(\sin x)^4}{4} + \ldots\ldots$$

$$= \left(x - \frac{x^3}{3!} + \ldots.\right) - \frac{1}{2}\left(x - \frac{x^3}{6} + \ldots.\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6} + \ldots.\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{6} + \ldots.\right)^4 + \ldots..$$

$$= x - \frac{x^3}{6} + \ldots.. - \frac{1}{2}\left(x^2 - \frac{2x^4}{6} + \ldots.\right) + \frac{1}{3}(x^3 + \ldots.) - \frac{1}{4}(x^4 + \ldots.) + \ldots.$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots..$$

(e) Show that every square matrix can be uniquely expressed as P + *i*Q, where P and Q are Hermitian matrices. **(4)**

Solution :

Let A be a square matrix of order 'n'.

We know that A $= \left(\frac{A+A^*}{2}\right)+\left(\frac{A-A^*}{2}\right)$

$= \left(\frac{A+A^*}{2}\right)+i\left(\frac{A-A^*}{2i}\right)$

$\therefore$ A = P + iQ (say) .....(i)

where $P = \frac{A+A^*}{2}$ and $Q = \frac{A-A^*}{2i}$

We shall show that P and Q are Hermitian.

$P^* = \left(\frac{A+A^*}{2}\right)^*$

$= \frac{1}{2}(A^* + (A^*)^*)$ $[\because (A+B)^* = A^* + B^*]$

$= \frac{1}{2}(A^* + A)$ $[\because (A^*)^* = A]$

$= P$

$\therefore$ P is a Hermitian matrix.

$Q^* = \left(\frac{A-A^*}{2i}\right)^*$

$= \overline{\left(\frac{1}{2i}\right)}\left(A-A^*\right)^*$ $\left\{\because (kA)^* = \bar{k}A^*\right\}$

$= \frac{-1}{2i}(A^* - (A^*)^*)$ $\left\{\because (A-B)^* = A^* - B^*\right\}$

$= \frac{-1}{2i}(A^* - A)$ $[\because (A^*)^* = A]$

$= \frac{1}{2i}(A - A^*)$

$= Q$

$\therefore$ Q is a Hermitian matrix.

Thus A is expressed as P + iQ where P and Q are Hermitian matrices.

(f) Find n^{th} order derivative of $\dfrac{x^2+4}{(x-1)^2(2x+3)}$. **(4)**

Solution :

$$y = \frac{x^2+4}{(x-1)^2(2x+3)}$$

Let $\dfrac{x^2+4}{(x-1)^2(2x+3)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{2x+3}$

$$= \frac{A(x-1)(2x+3) + B(2x+3) + C(x-1)^2}{(x-1)^2(2x+3)}$$

$\therefore \quad x^2+4 = A(x-1)(2x+3) + B(2x+3) + C(x-1)^2$

Put $x = 1$: $5 = 5B \quad \therefore \quad B = 1$

Put $x = -\dfrac{3}{2}$: $\dfrac{9}{4} + 4 = C\left(-\dfrac{3}{2} - 1\right)^2 \quad \therefore \dfrac{25}{4} = \dfrac{25}{4}C \quad \therefore \quad C = 1$

Compare coefficient of x^2

$\therefore \quad 1 = 2A + C \quad \therefore A = 0$

$$\therefore \quad y = \frac{1}{(x-1)^2} + \frac{1}{2x+3}$$

$$\therefore \quad y_n = \frac{d^n}{dx^n}\left[\frac{1}{(x-1)^2}\right] + \frac{d^n}{dx^n}\left[\frac{1}{2x+3}\right]$$

$$= \frac{(-1)^n 1^n (2+n-1)!}{(2-1)!(x-1)^{2+n}} + \frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}} = \frac{(-1)^n (n+1)!}{(x-1)^{n+2}} + \frac{(-1)^n n!.2^n}{(2x+3)^{n+1}}$$

$$y_n = (-1)^n.n!\left[\frac{(n+1)}{(x-1)^{n+2}} + \frac{2^n}{(2x+3)^{n+1}}\right]$$

2. (a) Show that the roots of the equation $(x+1)^6 + (x-1)^6 = 0$ are given by

$- i \cot\left[\dfrac{(2k+1)\pi}{12}\right]$, $k = 0, 1, 2, 3, 4, 5$ **(6)**

Solution :

$(x+1)^6 + (x-1)^6 = 0$

$ky\ (x-1)^6$

$\therefore \quad \left(\frac{x+1}{x-1}\right)^6 + 1 = 0$

$\therefore \quad \left(\frac{x+1}{x-1}\right)^6 = -1$

$$= \cos\pi + i\sin\pi = \cos(2k+1)\,\pi + i\sin(2k+1)\,\pi$$

where k = 0, 1, ..., 5.

$\therefore \quad \frac{x+1}{x-1} = \cos(2k+1)\,\frac{\pi}{6} + i\sin(2k+1)\,\frac{\pi}{6}$

$$= \cos\theta + i\sin\theta \text{ where } \theta = (2k+1)\,\frac{\pi}{6}$$

$$= e^{i\theta}$$

$\therefore$ By comp....

$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i\theta}+1}{e^{i\theta}-1}$$

$\therefore \quad x = \frac{e^{i\theta}+1}{e^{i\theta}-1}$

$\therefore \quad x = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$

$$= \frac{(1+\cos\theta)+i\sin\theta}{-(1-\cos\theta)+i\sin\theta} = \frac{2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2}\; i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} = \frac{i\cos\frac{\theta}{2}}{i^2\sin\frac{\theta}{2}} = -i\cot\frac{\theta}{2}$$

$$= -i\cot\left[\frac{(2k+1)\pi}{12}\right],\ k = 0, 1, ...5.$$

(b) Reduce the following matrix into normal form and find its rank $\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$. **(6)**

Solution :

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$= \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

$R_5 + R_4$

$$= \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - C_1,\ C_4 - 2C_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(-1)\ R_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 0 & 5 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 - 3R_2,\ R_4 - 4R_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & -13 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 - R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - C_2,\ C_4 - 3C_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\left(-\frac{1}{3}\right) C_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 + 14C_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Normal form}$$

$\therefore \quad P(A) = 3$

(c) State and prove the Euler's theorem for a homogeneous function in two variables.

Hence find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ if $u = \frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}}$. **(8)**

Solution :

If u is a homogeneous function in x and y of degree 'n' then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Proof :

$\because$ u is a homogeneous function in x and y of degree n.

$\therefore$ $u = x^n f\left(\frac{y}{x}\right)$ .....(i)

Differentiate (i) partially w.r.t. x,

$$\therefore \frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right).nx^{n-1}$$

$$= nx^{n-1}f\left(\frac{y}{x}\right) - y\ x^{n-2}\ f'\left(\frac{y}{x}\right)$$

Multiplying by x on both the sides,

$$x\frac{\partial u}{\partial x} = nx^n\ f\left(\frac{y}{x}\right) - yx^{n-1}f'\left(\frac{y}{x}\right) \quad .....(ii)$$

Differentiate (i) partially w.r.t. y

$$\therefore \frac{\partial u}{\partial y} = x^n\ f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right)$$

Multiplying by y on both the sides,

$$y\frac{\partial u}{\partial y} = yx^{n-1}f'\left(\frac{y}{x}\right) \quad .....(iii)$$

(ii) + (iii) $\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) - y\ x^{n-1}\ f'\left(\frac{y}{x}\right) + y\ x^{n-1}\ f'\left(\frac{y}{x}\right)$

$$= nx^n\ f\left(\frac{y}{x}\right)$$

$$= nu \quad \text{[using (i)]}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

$$u = \frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}} = \frac{x\sqrt{\frac{y}{x}}}{\sqrt{x}\left(1 + \sqrt{\frac{y}{x}}\right)} = x^{\frac{1}{2}} f\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function in x and y degree ½.

$\therefore$ By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = \frac{1}{2}\left[x^{\frac{1}{2}}\ f\left(\frac{y}{x}\right)\right] = \frac{1}{2}\ \frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}}$

3. (a) Test for consistency and solve if consistent

$$x_1 - 2x_2 + x_3 - x_4 = 2;\ x_1 + 2x_2 + 2x_4 = 1;\ 4x_2 - x_3 + 3x_4 = -1$$ **(6)**

Solution :

We have $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 2 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

By $R_2 - R_1$ $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

By $R_3 - R_2$ $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ .... (i)

The rank of coefficient matrix $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is 2 and rank of augmented matrix

$\begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 0 & 4 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is also 2.

Hence the equations are consistent.

But $r = 2 < 4$.

Hence, the number of parameters = 4 – 2 = 2.

Hence, equation have doubly infinite solutions.

Now, from (i)

$$x_1 + 2x_2 + 3x_3 + 2x_4 = 2,\ 4x_2 - x_3 + 3x_4 = -1$$

Put $x_3 = t_1$ and $x_4 = t_2$

Hence, $4x_2 = t_1 - 3t_2$ and $x_1 = 2[(t_1 - 3t_2)/4] + 3t_1 + 2t_2$

Hence, $x_2 = \dfrac{(t_1 - 3t_2)}{4}$ and $x_1 = \left|\dfrac{(t_1 - 3t_2)}{2}\right| + 3t_1 + 2t_2$

(b) Find all the stationary values of $x^3 + 3xy - 15x^2 - 15y^2 + 72x$. **(6)**

Solution :

$$F(x, y) = x^3 + 3xy - 15x^2 - 15y^2 + 72x$$

Step 1 :

$f_x = 3x^2 + 3y - 30x + 72$ and $f_y = 3x - 30y$

$r = f_{xx} = 6x - 30$, $t = f_{yy} = -30$ and $s = f_{xy} = 3$

Step 2 :

We now solve, $f_x = 0, f_y = 0$ as a simultaneous equations.

$\therefore \quad 3x^2 + 3y - 30x + 72 = 0$ and $3x - 30y = 0$;hence $x = 10y$

To eliminate x, we put $x = 10y$ in second equation

$\therefore \quad 100y^2 - 99y + 24 = 0$

On solving,

$\therefore \quad y = 0.565$ or $y = 0.424$

When $y = 0.565$, $x = 5.65$ and

when $y = 0.424$, $x = 424$

Step 3 :

For $x = 5.65$, $y = 0.565$,

$r = f_{xx} = 3.9$

$t = f_{yy} = -30$

$s = f_{xy} = 3$

Now, $rt - s^2 < 0$. Hence, we reject this pair.

When $x = 4.24$, $y = 0.424$,

$r = f_{xx} = -4.56$

$t = f_{yy} = -30$

$s = f_{xy} = 3$

Now, $rt - s^2 > 0$. Hence, we accept the pair.

***x* = 4.24 and *y* = 0.424**

(c) If $\tan\left(\frac{\pi}{4} + iv\right) = re^{i\theta}$, show that **(8)**

(i) $r = 1$ (ii) $\tan\theta = \sin h2v$ (iii) $\tan hv = \tan\frac{\theta}{2}$

Solution :

$\tan\left(\frac{\pi}{4} + iv\right) = re^{i\theta}$ (given) .... (1)

Consider $\tan\left(\frac{\pi}{4}+iv\right)=\dfrac{2\sin\left(\frac{\pi}{4}+iv\right)\cos\left(\frac{\pi}{4}-iv\right)}{2\cos\left(\frac{\pi}{4}+iv\right)\cos\left(\frac{\pi}{4}-iv\right)}$

$$=\frac{\sin\left[\left(\frac{\pi}{4}+iv\right)+\left(\frac{\pi}{4}-iv\right)\right]+\sin\left[\left(\frac{\pi}{4}+iv\right)-\left(\frac{\pi}{4}-iv\right)\right]}{\cos\left[\left(\frac{\pi}{4}+iv\right)+\left(\frac{\pi}{4}-iv\right)\right]+\cos\left[\left(\frac{\pi}{4}+iv\right)-\left(\frac{\pi}{4}-iv\right)\right]}$$

$$=\frac{\sin\frac{\pi}{2}+\sin(2iv)}{\cos\frac{\pi}{2}+\cos(2iv)}=\frac{1+i\sinh 2v}{\cosh\,(2v)}=\frac{1}{\cosh 2v}+i\,\frac{\sinh\,2v}{\cosh 2v}=\text{sech}\,2v+i\tanh\,2v$$

$\therefore$ From (1)

$\text{sech}\,2v+i\tanh\,2v=r(\cos\theta+i\sin\theta)$

$\therefore$ $\text{sech}\,2v=r\cos\theta$ ... (1)

$\tanh\,2v=r\sin\theta$ ... (2)

$(1)^2+(2)^2\;\Rightarrow$

$r^2=\text{sech}^2\,2v+\tanh^2 2v=1$

$\therefore\;r=1$

$\frac{(2)}{(1)}\Rightarrow\frac{r\sin\theta}{r\cos\theta}=\frac{\tanh\,2v}{\text{sech}\,2v}$

$\therefore\;\tan\theta=\sinh 2v$

Now $2v=\sinh^{-1}(\tan\theta)$

$$=\log(\tan\theta+\sqrt{1+\tan^2\theta})=\log(\tan\theta+\sec\theta)=\log\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$=\log\left[\frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right]=\log\left[\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]=\log\left[\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right]$$

$$=2\tanh^{-1}\left(\tan\frac{\theta}{2}\right)\qquad\left\{\because\;\tanh^{-1}z=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right\}$$

$\therefore\;v=\tanh^{-1}\left(\tan\frac{\theta}{2}\right)\qquad\therefore\;\tanh v=\tan\frac{\theta}{2}$

4. (a) If $x = u + e^{-v} \sin u$, $y = v + e–^{u} \cos u$, find $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ by using Jacobian. **(6)**

Solution :

We have $f_1 = x - u - e^{-v} \sin u$; $f_2 = y - v - e^{-v} \cos u$.

$$\therefore \quad \frac{\partial u}{\partial y} = -\left[\frac{\partial(f_1, f_2)}{\partial(y,v)} \bigg/ \frac{\partial(f_1, f_2)}{\partial(u,v)}\right] \text{ and } \frac{\partial v}{\partial x} = -\left[\frac{\partial(f_1, f_2)}{\partial(u,x)} \bigg/ \frac{\partial(f_1, f_2)}{\partial(u,v)}\right]$$

Now, $\frac{\partial(f_1, f_2)}{\partial(y,v)} = \begin{vmatrix} 0 & 1 \\ e^{-v}\sin u & -1+e^{-v}\cos u \end{vmatrix} = -\, e^{-v} \sin u$

$$\frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} -1-e^{-v}\cos u & e^{-v}\sin u \\ e^{-v}\sin u & -1+e^{-v}\cos u \end{vmatrix}$$

$$= 1 + e^{-v}\cos u - e^{-v}\cos u - e^{-2v}\cos^2 u - e^{-2v}\sin^2 u$$

$$= 1 - e^{-2v}(\sin^2 u + \cos^2 u) = 1 - e^{-2v}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{e^{-v}\sin u}{1-e^{-2v}}$$

Also, $\frac{\partial(f_1, f_2)}{\partial(u,x)} = \begin{vmatrix} -1-e^{-v}\cos u & 1 \\ e^{-v}\sin u & 0 \end{vmatrix} = -\, e^{-v} \sin u$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{e^{-v}\sin u}{1-e^{-2v}}$$

(b) Considering only the principal value, if $(1 + i \tan\alpha)^{1 + i\tan\beta}$ is real, prove that its values is $(\sec\alpha)^{\sec^2\beta}$. **(6)**

Solution :

Let $z = (1+i\tan\alpha)^{1+i\tan\beta}$

Taking logarithms of both sides,

$$\log z = (1+i\tan\beta)\,\log(1+i\tan\alpha)$$

$$= (1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha) + i\tan^{-1}\tan\alpha\right] = (1+i\tan\beta)\,[\log\sec\alpha + i\alpha]$$

$$\therefore \quad \log z = (\log\sec\alpha - \alpha\tan\beta) + i(\alpha + \tan\beta\log\sec\alpha) = x + iy \text{ say}$$

where $x = \log\sec\alpha - \alpha\tan\beta$ and $y = \alpha + \tan\beta\,\log\sec\alpha$ .... (i)

Now, $z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$

Since by data z is real $e^x \sin y = 0 \quad \therefore\ y = 0 \qquad \therefore\ \cos y = 1$

$$\therefore \quad z = e^x \cos y = e^x = e^{\log\sec\alpha - \alpha\tan\beta}$$

$\therefore \quad z = e^{\log\sec\alpha} . e^{-\alpha\tan\beta} = \sec\alpha . e^{-\alpha\tan\beta}$ .... (ii)

But since y = 0, from (i), $\alpha + \tan\beta \log\sec\alpha = 0$

$\therefore \quad -\alpha = \tan\beta \log \sec\alpha$

$\therefore \quad -\alpha\tan\beta = \tan^2\beta . \log\sec\alpha = \log(\sec\alpha)^{\tan^2\beta}$

$\therefore \quad e^{-\alpha\tan\beta} = (\sec\alpha)^{\tan^2\beta}$

$\therefore \quad z = \sec\alpha . (\sec\alpha)^{\tan^2\beta} = (\sec\alpha)^{(1+\tan^2\beta)}$ .... [From (ii)]

$= (\sec\alpha)^{\sec^2\beta}$

(c) Solve the system of linear equation by Crout's method $x - y + 2z = 2$; $3x + 2y - 3z = 2$; $4x - 4y + 2z = 2$. **(8)**

Solution :

(i) Let us write the system as

$$AX = B \text{ i.e., } \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -3 \\ 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(ii) Now, we write LU = A and find L and U.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -3 \\ 4 & -4 & 2 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -3 \\ 4 & -4 & 2 \end{bmatrix}$$

Equating the corresponding elements in the two matrices, we get

(1) $l_{11} = 1,\ l_{11}u_{12} = -1 \quad \therefore\ u_{12} = -1,\ l_{11}u_{13} = 2 \quad \therefore\ u_{13} = 2$

(2) $l_{21} = 3,\ l_{21}u_{12} + l_{22} = 2 \quad \therefore\ l_{22} = 2 - (3)(-1) = 2 + 3 = 5 \quad \therefore\ l_{22} = 5$

$l_{21}u_{13} + l_{22}u_{23} = -3$

$\therefore \quad l_{22}u_{23} = -3 - l_{21}u_{13} = -3 - (3)(2) = -9 \quad \therefore\ u_{23} = -\dfrac{9}{5}$

(3) $l_{31} = 4,\ l_{31}u_{12} + l_{32} = -4 \quad \therefore\ l_{32} = -4 - l_{31}u_{12}$

$\therefore \quad l_{32} = -4 - (4)(-1) = 0 \quad \therefore\ l_{32} = 0$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$

$$4(2) + 0\left(-\frac{9}{5}\right) + l_{33} = 2 \quad \therefore \; l_{33} = 2 - 8 = -6 \quad \therefore \; l_{33} = -6$$

$$\therefore \quad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 0 & -6 \end{bmatrix}$$

and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -9/5 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) Now, LV = B, gives $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 0 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$\therefore \quad v_1 = 2, \; 3v_1 + 5v_2 = 2 \qquad \therefore \; 5v_2 = 2 - 6$

$\therefore \quad 5v_2 = -4 \qquad \therefore \; v_2 = -\frac{4}{5}$

$4v_1 + 0v_2 - 6v_3 = 2$

$\therefore \quad -6v_3 = 2 - 4v_1 = 2 - 8 = -6 \qquad \therefore \; v_3 = 1$

(iv) Now, UX = V, gives $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -9/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4/5 \\ 1 \end{bmatrix}$

$\therefore \quad x - y + 2z = 2, \; y - \frac{9}{5}z = -\frac{4}{5}, \; z = 1$

$\therefore \quad y = -\frac{4}{5} + \frac{9}{5}z = \frac{5}{5} = 1$

$\therefore \quad x - y + 2z = 2 \qquad \therefore \; x - 1 + 2 = 2 \qquad \therefore \; x = 1$

$\therefore \quad x = 1, y = 1, z = 1.$

5. (a) Expand $\cos^7 \theta$ in a series of cosines of multiples of θ. **(6)**

Solution :

Let $x = \cos\theta + i\sin\theta$ $\qquad$ $x^n = \cos n\theta + i\sin n\theta$

$\therefore \quad \frac{1}{x} = \cos\theta - i\sin\theta$ $\qquad$ $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore \quad x + \frac{1}{x} = 2\cos\theta$ $\qquad$ $\therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta$

$\therefore \quad x - \frac{1}{x} = 2i\sin\theta \qquad\qquad \therefore \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$\sin^7\theta = \left[\frac{1}{2i}\left(x-\frac{1}{x}\right)\right]^7 = \frac{1}{2^7 i^7}\left(x-\frac{1}{x}\right)^7 = \frac{1}{2^7(-i)}\left(x-\frac{1}{x}\right)^7$$

$$= \frac{-1}{2^7 i}\left[{}^7C_0x^7 - {}^7C_1x^6\frac{1}{x} + {}^7C_2x^5\frac{1}{x^2} - {}^7C_3x^4\frac{1}{x^3} + {}^7C_4x^3\frac{1}{x^4} - {}^7C_5x^2\frac{1}{x^5} + {}^7C_6x\frac{1}{x^6} - {}^7C_7\frac{1}{x^7}\right]$$

$$= -\frac{1}{2^7 i}\left[x^7 - 7x^5 + 21x^3 - 35x + 35\frac{1}{x} - 21\frac{1}{x^3} + 7\frac{1}{x^5} - \frac{1}{x^7}\right]$$

$$= -\frac{1}{2^7 i}\left[\left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)\right]$$

$$= -\frac{1}{2^7 i}[2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)]$$

$$= -\frac{1}{2^6}[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta]$$

(b) Evaluate $\lim_{x\to 0}\left[\frac{1}{x^2} - \cot^2 x\right]$. **(6)**

Solution :

$$l = \lim_{x\to 0}\left(\frac{1}{x^2} - \cot^2 x\right) = \lim_{x\to 0}\left(\frac{\tan^2 x - x^2}{x^2\tan^2 x}\right)$$

$$= \lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4}\cdot\frac{x^2}{\tan^2 x} = \lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4}\cdot\lim_{x\to 0}\frac{1}{\left(\frac{\tan x}{x}\right)^2}$$

$$= \lim_{x\to 0}\frac{\left(x + \frac{x^3}{3} + \ldots\ldots\right)^2 - x^2}{x^4}\cdot\frac{1}{1} = \lim_{x\to 0}\frac{x^2 + \frac{2x^4}{3} + \frac{x^6}{9} + \ldots\ldots - x^2}{x^4}$$

$$= \lim_{x\to 0}\frac{\frac{2}{3}x^4 + \text{higher powers of } x}{x^4} = \lim_{x\to 0}\frac{x^4\left(\frac{2}{3} + \text{terms containing } x\right)}{x^4}$$

$$= \frac{2}{3} + 0 = \frac{2}{3}.$$

(c) If $y = (\sin^{-1} x)^2$, obtain $y_n(0)$. **(8)**

Solution :

We have $y = (\sin^{-1} x)^2$ ... (1)

$\therefore \quad y_1 = 2(\sin^{-1} x).\dfrac{1}{\sqrt{1-x^2}}$ ... (2)

$\therefore \quad \sqrt{1-x^2}\,.\,y_1 = 2\sin^{-1} x$

Differentiating again,

$$\sqrt{1-x^2}\,.\,y_2 - y_1.\frac{x}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$\therefore \quad (1-x^2)y_2 - x.y_1 = 2$ ... (3)

By Leibnitz's theorem,

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - [xy_{n+1} + n(1)y_n] = 0$$

$\therefore \quad (1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - n^2 y_n = 0$ ... (4)

Putting $x = 0$, $y_{n+2}(0) = n^2 y_n(0)$ ... (5)

Putting $x = 0$ in (1), (2) and (3), we get,

$y(0) = (0)$, $y_1 = (0) = 0$, $y_2(0) = 2$.

Putting $n = 1, 2, 3, \ldots\ldots$, in (5), we get,

$y_3(0) = 1^2.y_1(0) = 0,\ y_5(0) = 3^2.y_3(0) = 0,$

$y_7(0) = 5^2.y_5(0) = 0, \ldots\ldots.,\ y_n(0) = 0$ if n is odd

Putting $n = 2, 4, 6, \ldots\ldots$, in (5), we get,

$y_4(0) = 2^2.y_2(0) = 2^2.2,\ 6_6(0) = 4^2.\ y_4(0) = 4^2.2^2.2$

$y_n(0) = 2.2^2.4^2.6^2 \ldots\ldots.(n-2)^2$ in n is even and $n \neq 2$.

6. (a) Show that the vectors are linearly dependent and find the relation between them :

$X_1 = [1, 2, -1, 0]$, $X_2 = [1, 3, 1, 2]$, $X_3 = [4, 2, 1, 0]$, $X_4 = [6, 1, 0, 1]$ **(6)**

Solution :

Consider the matrix equation

$k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = 0$

$\therefore \quad k_1[1, 2, -1, 0] + k_2[1, 3, 1, 2] + k_3[4, 2, 1, 0] + k_4[6, 1, 0, 1] = [0, 0, 0, 0]$

$\therefore \quad k_1 + k_2 + 4k_3 + 6k_4 = 0$

$2k_1 + 3k_2 + 3k_3 + k_4 = 0$

$-k_1 + k_2 + k_3 = 0$

$2k_2 + k_4 = 0$

which can be written as

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 2R_1,\ R_3 + R_1$ $$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2,\ R_4 - R_3$ $$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_4 + \frac{5}{17} R_3$ $$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 0 & 55/17 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad k_1 + k_2 + 4k_3 + 6k_4 = 0$

$k_2 - 6k_3 + k_4 = 0$

$17k_3 + 28k_4 = 0$

$\frac{55}{17} k_4 = 0$

$\therefore \quad k_4 = 0,\ k_3 = 0,\ k_2 = 0,\ k_1 = 0$

Hence, the vectors are independent.

(b) If $\frac{x^2}{1+u} + \frac{y^2}{2+u} + \frac{z^2}{3+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right]$$ **(6)**

Solution :

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad \text{..... (i) (given)}$$

Differentiate (i) partially w.r.t. x

$$\therefore \quad \frac{(a^2+u)(2x)-x^2\cdot\frac{\partial u}{\partial x}}{(a^2+u)^2}+y^2\left(\frac{-1}{(b^2+u)^2}\right)\cdot\frac{\partial u}{\partial x}+z^2\left(\frac{-1}{(c^2+u)^2}\right)\cdot\frac{\partial u}{\partial x}=0$$

$$\therefore \quad \frac{2x}{(a^2+u)}-\frac{x^2}{(a^2+u)^2}\cdot\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\cdot\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\cdot\frac{\partial u}{\partial x}=0$$

$$\therefore \quad u_x=\frac{2x}{a^2+u}\div\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

Let $t=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$

$$\therefore \quad t.u_x=\frac{2x}{a^2+u} \qquad \text{..... (ii)}$$

The given function is symmetric function of *ax*, *by*, *cz*

$$\therefore \quad t.u_y=\frac{2y}{b^2+u} \qquad \text{.....(iii)}$$

$$t.u_z=\frac{2z}{c^2+u} \qquad \text{.....(iv)}$$

Squaring and adding (ii), (iii), (iv)

$$t^2\left[(u_x)^2+(u_y)^2+(u_z)^2\right]=\frac{4x^2}{(a^2+u)^2}+\frac{4y^2}{(b^2+u)^2}+\frac{4z^2}{(c^2+u)^2}=4t$$

$$\therefore \quad (u_x)^2+(u_y)^2+(u_z)^2=\frac{4}{t} \qquad \text{.....(v)}$$

Multiply (ii) by *x*, (iii) by *y*, (iv) by *z* and adding we get

$$2\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]=t\left[xu_x+yu_y+zu_z\right]$$

$$\therefore \quad xu_x+yu_y+zu_z=\frac{2}{t} \quad \text{[using (i)]}$$

$$\therefore \quad 2\left[xu_x+yu_y+zu_z\right]=\frac{4}{t} \qquad \text{.....(vi)}$$

From (v) & (vi) we get the required result.

(c) Fit a second degree parabolic curve to the following data **(8)**

[redacted]

Solution :

Since the values of x are odd and are equally spaced we change X to x by the X = x – 5 and put y = Y.

Let the equation of the parabola be $Y = a + bX + cX^2$. Then the normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$

$$\Sigma X^2 Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

Calculations of ΣX, ΣX^2 etc.

[redacted]

Now, we have

$\Sigma Y = NA + B\Sigma X \quad \therefore \quad 11.5835 = 5A + 20B$

$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \therefore \quad 47.1254 = 20A + 90B$

Solving these equations, we get,

B = 0.07914 and A = 2.0001

Hence, b = antilog 0.7914 = 1.199, a = antilog 2.0001 = 100

$\therefore$ The law is $y = 100\,(1.199)^x$.