VIJAY SHEKHAR ACADEMY

Coaching. Excelling. Leading.

Max. Marks : 80 | **Applied Mathematics - I** | Date :

Duration : 3 Hours | **March - 2013** | Test No.

Instructions : (i) Question No. 1 is compulsory.
(ii) Attempt any three questions from the remaining five.
(iii) Figures to the right indicate full marks.

1. (a) If $\cos hx = \sec \theta$, prove that $x = \log(\sec\theta + \tan\theta)$ **(3)**

Solution :

(b) If $u = \log(x^2 + y^2)$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. **(3)**

Solution :

Differentiate u part w.r.t. y

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

Differentiate part w.r.t. x

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2y\left[\frac{-1}{(x^2 + y^2)^2}\right]2x$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{(x^2 + y^2)^2} \qquad \text{.... (i)}$$

Differentiate u part w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{2x}{(x^2 + y^2)}$$

Differentiate part w.r.t. y

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2}\right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2x\left[\frac{-1}{(x^2 + y^2)^2}\right](2xy)$$

$$\frac{\partial^2 u}{\partial y \partial u} = -\frac{4xy}{(x^2 + y^2)^2} \qquad \text{.... (ii)}$$

$\therefore$ From (i) and (ii)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(c) If $x = r\cos\theta$, $y = r\sin\theta$. Find $\dfrac{\partial(x, y)}{\partial(r, \theta)}$. **(3)**

Solution :

(d) Expand $\log(1 + x + x^2 + x^3)$ in powers of x upto x^8. **(3)**

Solution :

$$\log(1 + x + x^2 + x^3) = \log\left[\frac{1(1-x^4)}{1-x}\right]$$

$$= \log(1-x^4) - \log(1-x)$$

$$= \left(-x^4 - \frac{x^8}{2} - ......\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \frac{x^8}{8} - .....\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + ...$$

(e) Show that every square matrix can be uniquelly expressed as sum of a symmetric and a Skew-symmetric matrix. **(4)**

Solution :

Let A be a square matrix of order 'n'.

We know that $A = \left(\dfrac{A + A^T}{2}\right) + \left(\dfrac{A - A^T}{2}\right)$

$\therefore$ $A = B + C$ (say) .....(i)

where $B = \dfrac{A + A^T}{2}$ and $C = \dfrac{A - A^T}{2}$

We shall show that B is symmetric and C is skew symmetric.

$$B^T = \left(\frac{A + A^T}{2}\right)^T$$

$$= \frac{1}{2}(A^T + (A^T)^T) \quad [\because (A + B)^T = A^T + B^T]$$

$$= \frac{1}{2}(A^T + A) \quad [\because (A^T)^T = A]$$

$$= B$$

$\therefore$ B is a symmetric matrix.

$$C^T = \left(\frac{A - A^T}{2}\right)^T$$

$$= \frac{1}{2}(A^T - (A^T)^T) \quad [\because (A-B)^T = A^T - B^T]$$

$$= \frac{1}{2}(A^T - A) \quad [\because (A^T)^T = A]$$

$$= -\frac{1}{2}(A - A^T)$$

$$= -C$$

$\therefore$ C is a skew symmetric matrix.

Thus A is expressed as a sum of a symmetric and skew symmetric matrix.

(f) Find n^{th} order derivative of $y = \cos x.\cos 2x.\cos 3x$. **(4)**

Solution :

$y = \cos x \cos 2x \cos 3x$

$$= \frac{1}{2}[2\cos 2x \cos x] \cos 3x = \frac{1}{2}[\cos 3x + \cos x] \cos 3x$$

$$= \frac{1}{4}[(2\cos 3x \cos 3x) + (2\cos 3x \cos x)] = \frac{1}{4}[(\cos 6x + \cos 0) + (\cos 4x + \cos 2x)]$$

$$= \frac{1}{4}[\cos 6x + 1 - \cos 4x + \cos 2x]$$

$$= \frac{1}{4}\left[6^n \cos\left(6x + \frac{n\pi}{2}\right) + 0 + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$$

2. (a) Solve the equation $x^6 - i = 0$ **(6)**

Solution :

Consider $x^4 + 1 = 0$

$$\therefore \quad x^4 = -1 = \cos\pi + i\sin\pi = \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\therefore \quad x = \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}, \; k = 0, 1, 2, 3$$

When $k = 0 : x_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$k = 1 : \; x_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$$k = 2: \; x_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$k = 3: \; x_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \cos\left(2\pi - \frac{\pi}{4}\right) + i\sin\left(2\pi - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Now consider $x^6 - i = 0$

$$\therefore \quad x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \cos\left(2k + \frac{1}{2}\right)\pi + i\sin\left(2k + \frac{1}{2}\right)\pi$$

where $k = 0, 1, 2, 3, 4, 5$

$$\therefore \quad x = \cos\left(2k + \frac{1}{2}\right)\frac{\pi}{6} + i\sin\left(2k + \frac{1}{2}\right)\frac{\pi}{6}$$

When $k = 0: x_1 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$

When $k = 1: x_2 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

When $k = 2: x_3 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

When $k = 3: x_4 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

When $k = 4: x_5 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

When $k = 5: x_6 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$$= \cos\left(2\pi - \frac{\pi}{4}\right) + i\sin\left(2\pi - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$\therefore$ Common roots are $-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

(b) Reduce matrix A to normal form and find its rank where

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$ **(6)**

Solution :

(c) State and prove Euler's theorem for a homogeneous function in two variables and hence find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ where $u = \dfrac{\sqrt{x}+\sqrt{y}}{x+y}$. **(8)**

Solution :

3. (a) Determine the values of λ so that the equations $x + y + z = 1$; $x + 2y + 4z = \lambda$; $x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case. **(6)**

Solution :

(b) Find the stationary values of $x^3 + y^3 - 3axy$, $a > 0$. **(6)**

Solution :

(c) Separate into real and imaginary parts $\tan^{-1}(e^{i\theta})$. **(8)**

Solution :

Let $\tan^{-1}(e^{i\theta}) = x + iy$ .....(i)

$\therefore \tan(x + iy) = e^{i\theta}$ and $\tan(x - iy) = e^{-i\theta}$

Now, $\tan 2x = \tan[(x + iy) + (x - iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy).\tan(x-iy)}$$

$$= \frac{e^{i\theta}+e^{-i\theta}}{1-e^{i\theta}.e^{-i\theta}} = \frac{e^{i\theta}+e^{-i\theta}}{1-1} = \infty = \tan\frac{\pi}{2}$$

$$\therefore \quad 2x = n\pi + \frac{\pi}{2}$$

$$\therefore \quad x = \frac{n\pi}{2}+\frac{\pi}{4}$$

Also, $\tan(2iy) = \tan[(x + iy) - (x - iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy).\tan(x-iy)}$$

$$= \frac{e^{i\theta}-e^{-i\theta}}{1+e^{i\theta}.e^{-i\theta}} = \frac{e^{i\theta}-e^{-i\theta}}{2} = i\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$$

$\therefore \tan(2iy) = i \sin\theta$

$\therefore$ i $\tanh 2y = i \sin\theta$

$\therefore \tanh 2y = \sin\theta$

$\therefore \quad 2y = \tanh^{-1}(\sin\theta)$

$$= \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{1}{2}\log\left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{1-\cos\left(\frac{\pi}{2}-\theta\right)}\right]$$

$$= \frac{1}{2}\log\left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right] = \frac{1}{2}\log\cot^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \log\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]^{-1}$$

$$= -\log\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$\therefore \quad y = -\frac{1}{2}\log\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$\therefore \quad (1) \Rightarrow \tan^{-1}(e^{i\theta}) = \left(\frac{n\pi}{2}+\frac{\pi}{4}\right) - \frac{i}{2}\log\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

4. (a) If $x = u\cos v$, $y = u\sin v$, prove that $\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = 1$. **(6)**

Solution :

(b) If $\tan[\log(x + iy)] = a + ib$, prove that $\tan[\log(x^2+y^2)] = \frac{2a}{(1-a^2-b^2)}$

where $a^2+b^2 \neq 1$ **(6)**

Solution :

$\tan[\log(x+iy)] = a+ib$

$\therefore \quad \log(x + iy) = \tan^{-1}(a + ib)$

Let $\tan^{-1}(a + ib) = \alpha + i\beta$

$\therefore \quad \log(x + iy) = \alpha + i\beta$

$$\therefore \quad \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) = \alpha + i\beta$$

$$\therefore \quad \frac{1}{2}\log(x^2+y^2) = \alpha \text{ and } \tan^{-1}\left(\frac{y}{x}\right) = \beta$$

$\therefore \quad 2\alpha = \log(x^2+y^2)$

Now consider $\tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$

$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\tan(\alpha - i\beta)} = \frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}$$

$$\therefore \quad \tan[\log(x^2 + y^2)] = \frac{2a}{1 - a^2 - b^2}$$

(c) Using Gauss-Siedel iteration method, solve **(8)**

$$10x_1 + x_2 + x_3 = 12$$

$$2x_1 + 10x_2 + x_3 = 13$$

$$2x_1 + 2x_2 + 10x_3 = 14$$

upto three iterations.

Solution :

We first write the equation as

$$x_1 = \frac{1}{10}(12 - x_2 - x_3) \qquad \text{... (1)}$$

$$x_2 = \frac{1}{10}(13 - 2x_1 - x_3) \qquad \text{... (2)}$$

$$x_3 = \frac{1}{10}(14 - 2x_1 - 2x_2) \qquad \text{... (3)}$$

(i) First Iteration :

We start with the approximation $x_2 = 0$, $x_3 = 0$ and then we get from (1)

$$\therefore \quad x_{1.1} = \frac{12}{10} = 1.2$$

We use this approximation to find x_2 i.e. we put $x_1 = 1.2$, $x_3 = 0$ in (2)

$$\therefore \quad x_{2.1} = \frac{1}{10}[13 - 2(1.2)] = 1.06$$

We use these values of x_1 and x_2 to find x_3

$$\therefore \quad x_{3.1} = \frac{1}{10}[14 - 2(1.2) - 2(1.06)] = 0.948$$

(ii) Second Iteration :

We use the latest values of x_2 and x_3 to find x_1 i.e., we put $x_2 = 1.06$, $x_3 = 0.948$ and the we get from (1)

$$\therefore \quad x_{1.2} = \frac{1}{10}[12 - 1.06 - 0.948] = 0.9992$$

We use this approximation to find x_2 i.e. we put $x_1 = 0.9992$, $x_3 = 0.948$ in (2)

$$\therefore \quad x_{2.2} = \frac{1}{10}[13 - 2(0.9992) - 0.948] = 1.00536$$

We use these values of x_1 and x_2 to find x_3

$$\therefore \quad x_{3.2} = \frac{1}{10}[14 - 2\,(0.9992) - 2(1.00536)] = 0.999088$$

(iii) Third Iteration :

We use the latest values of x_2 and x_3 to find x_1 i.e., we put $x_2 = 1.00536$, $x_3 = 0.999088$ and then we get from (1)

$$\therefore \quad x_{1.3} = \frac{1}{10}[12 - 1.00536 - 0.999088] = 0.9995552$$

We use this approximation to find x_2 i.e., we put $x_1 = 0.9995552$, $x_3 = 0.999088$ in (2)

$$\therefore \quad x_{2.3} = \frac{1}{10}[13 - 2(0.9995552) - 0.999088] = 1.00018$$

We use these values of x_1 and x_2 to find x_3

$$\therefore \quad x_{3.3} = \frac{1}{10}[14 - 2(0.9995552) - 2(1.00018)] = 1.00052$$

Since, the second and third iteration give the same values

$\boldsymbol{x_1 = 1, x_2 = 1, x_3 = 1}$

5. (a) Expand $\sin^7\theta$ in a series of sines of multiples of θ. **(6)**

Solution :

Let $x = \cos\theta + i\sin\theta$ $\qquad$ $x^n = \cos n\theta + i\sin n\theta$

$$\therefore \quad \frac{1}{x} = \cos\theta - i\sin\theta \qquad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore \quad x + \frac{1}{x} = 2\cos\theta \qquad \therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$\therefore \quad x - \frac{1}{x} = 2i\sin\theta \qquad \therefore \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$\sin^7\theta = \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^7 = \frac{1}{2^7 i^7}\left(x - \frac{1}{x}\right)^7 = \frac{1}{2^7(-i)}\left(x - \frac{1}{x}\right)^7$$

$$= \frac{-1}{2^7 i}\left[\begin{matrix} {}^7C_0x^7 - {}^7C_1x^6\frac{1}{x} + {}^7C_2x^5\frac{1}{x^2} - {}^7C_3x^4\frac{1}{x^3} + {}^7C_4x^3\frac{1}{x^4} - {}^7C_5x^2\frac{1}{x^5} \\ + {}^7C_6x\frac{1}{x^6} - {}^7C_7\frac{1}{x^7} \end{matrix}\right]$$

$$= -\frac{1}{2^7 i}\left[x^7 - 7x^5 + 21x^3 - 35x + 35\frac{1}{x} - 21\frac{1}{x^3} + 7\frac{1}{x^5} - \frac{1}{x^7}\right]$$

$$= -\frac{1}{2^7 i}\left[\left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)\right]$$

$$= -\frac{1}{2^7 i}[2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)]$$

$$= -\frac{1}{2^6}[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta]$$

(b) Evaluate $\lim_{x\to 0}\frac{(x^x - x)}{(x-1-\log x)}$. **(6)**

Solution :

(c) If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0$ **(8)**

Solution :

$$y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$$

$$\therefore \quad \frac{1}{m}y^{\frac{1}{m}-1}\frac{dy}{dx} - \frac{1}{m}y^{\frac{-1}{m}-1}\frac{dy}{dx} = 2$$

$$\therefore \quad \frac{dy}{dx}(y^{1/m} - y^{-1/m}) = 2my$$

$$\therefore \quad (y^{1/m} - y^{-1/m}) = \frac{2my}{\frac{dy}{dx}}$$

$$\because \quad (y^{1/m} + y^{-1/m})^2 - (y^{1/m} - y^{-1/m})^2 = 4$$

$$\therefore \quad 4x^2 - \frac{4m^2y^2}{\left(\frac{dy}{dx}\right)^2} = 4$$

$$\therefore \quad 4x^2\left(\frac{dy}{dx}\right)^2 - 4\left(\frac{dy}{dx}\right)^2 = 4m^2y^2$$

$$\therefore \quad (x^2-1)y_1^2 = m^2y^2$$

Differentiate again w.r.t. x

$$\therefore \quad (x^2 - 1)2y_1y_2 + y_1^2(2x) = m^2.2yy_1$$

$$\therefore \quad (x^2 - 1)y_2 + xy_1 - m^2y = 0$$

Differentiating 'n' times by Leibnitz's theorem we get

$$(x^2 - 1)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}2y_n + xy_{n+1} + n(1)y_n - m^2y_n = 0$$

$$\therefore \quad (x^2 - 1)y_{n+2} + (2n + 1)x\,y_{n+1} + (n^2 - m^2)y_n = 0$$

6. (a) Examine the following vectors for linear dependence / Independence.

$X_1 = (a, b, c)$, $X_2 = (b, c, a)$, $X_3 = (c, a, b)$ where $a + b + c \neq 0$. **(6)**

(b) If $z = f(x, y), x = e^u + e^{-v}, y = e^{-u} - e^v$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$ **(6)**

(c) Fit a straight line to the following data and estimate the production in the year 1957.

Year :	1951	1961	1971	1981	1991
Production in the thousand tons :	10	12	8	10	13

(8)