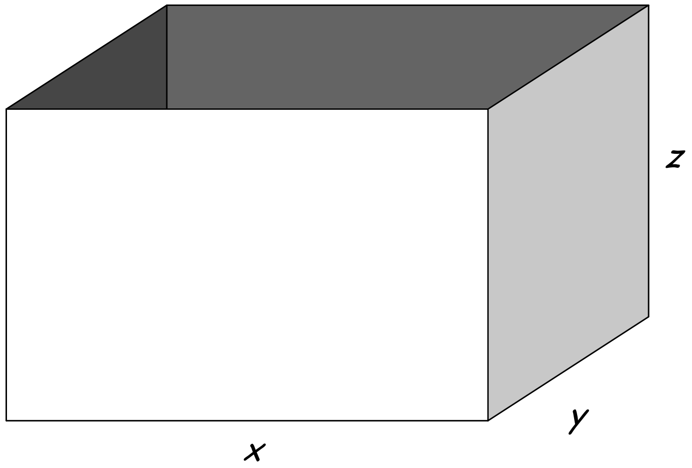
Complete Solutions to Examination Questions 15

1. Since we form an open top box, this means we have:



The surface area A is given by

$$A = xy + 2xz + 2yz \qquad (*)$$

We are given that the volume of the box is 3m^3 therefore $xyz = 3$. Making z the subject gives

$$z = \frac{3}{xy} \qquad (\dagger)$$

Substituting this into (*) yields

$$A = xy + 2\cancel{x}\frac{3}{\cancel{x}y} + 2\cancel{y}\frac{3}{x\cancel{y}}$$
$$= xy + \frac{6}{y} + \frac{6}{x}$$

Thus we have our required result.
To find which dimensions give minimum surface area we first find the stationary points of A. These are determined by partially differentiating the derived result and equating them to zero. We have

$$A = xy + \frac{6}{y} + \frac{6}{x} = xy + 6y^{-1} + 6x^{-1}$$

$$\frac{\partial A}{\partial x} = y + 0 - 6x^{-2} = 0 \text{ implies } y = \frac{6}{x^2}$$

$$\frac{\partial A}{\partial y} = x - 6y^{-2} = 0 \text{ implies } x - \frac{6}{y^2} = 0$$

Substituting $y = \frac{6}{x^2}$ into the bottom equation $x - \frac{6}{y^2} = 0$ gives

$$x - \frac{6}{\left(6/x^2\right)^2} = x - \frac{6}{36/x^4}$$
$$= x - \frac{x^4}{6} = \frac{x}{6}\left(6 - x^3\right) = 0$$

Solving the last equation $\frac{x}{6}\left(6 - x^3\right) = 0$ we have

$$x = 0, \quad x = 6^{1/3}$$

We cannot have $x = 0$ because this means we will not have a box. Hence $x = 6^{1/3} = 1.8171$.

Substituting this $x = 6^{1/3}$ into the above equation $y = \dfrac{6}{x^2}$ yields that

$$y = \frac{6}{\left(6^{1/3}\right)^2} = \frac{6^1}{6^{2/3}} = 6^{1-2/3} = 6^{1/3}$$

Hence $y = x = 6^{1/3} = 1.8171$. We need to check that these x and y values do indeed give us minimum surface area. *How?*

By using the second derivative test. We have

$$\frac{\partial A}{\partial x} = y - 6x^{-2}$$

$$\frac{\partial^2 A}{\partial x^2} = 0 - 6(-2)x^{-3} = 12x^{-3} = \frac{12}{x^3}$$

Similarly (or by examining the symmetry of the expression) from $\dfrac{\partial A}{\partial y} = x - 6y^{-2}$ we obtain

$\dfrac{\partial^2 A}{\partial y^2} = \dfrac{12}{y^3}$. *Which other second derivative do we need to find?*

The mixed partial derivative $\dfrac{\partial^2 A}{\partial x \partial y}$. This partial derivative is given by

$$\frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 6y^{-2}\right) = 1 - 0 = 1$$

Putting all these $\dfrac{\partial^2 A}{\partial x^2} = \dfrac{12}{x^3}$, $\dfrac{\partial^2 A}{\partial y^2} = \dfrac{12}{y^3}$ and $\dfrac{\partial^2 A}{\partial x \partial y} = 1$ into formula (15.11)

$$\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial y^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial y}\right)^2 = \left(\frac{12}{x^3}\right)\left(\frac{12}{y^3}\right) - 1^2 = \frac{144}{x^3 y^3} - 1 \qquad (**)$$

Substituting $y = x = 6^{1/3}$ into this (**) gives

$$\frac{144}{\left(6^{1/3}\right)^3\left(6^{1/3}\right)^3} - 1 = \frac{144}{36} - 1 > 0 \qquad [\text{Positive}]$$

We know from formula (15.11) that we have a minimum or maximum. Putting $x = 6^{1/3}$ into $\dfrac{\partial^2 A}{\partial x^2} = \dfrac{12}{x^3}$ gives $\dfrac{\partial^2 A}{\partial x^2} = \dfrac{12}{\left(6^{1/3}\right)^3} = \dfrac{12}{6} = 2 > 0$. Hence we have a minimum when $y = x = 6^{1/3}$.

What is the value of z?

Substituting $y = x = 6^{1/3}$ into above (†) equation which is $z = \dfrac{3}{xy}$ gives

$$z = \frac{3}{xy} = \frac{3}{6^{1/3}6^{1/3}} = \frac{3}{6^{2/3}} = 0.9086$$

Our dimensions for minimum surface area are $y = x = 1.817$ and $z = 0.909$ (correct to 3dp).

2. We need to find the critical points of the given function $f(x, \quad y) = y^3 - x^3 + 3xy + 1$.
To determine the stationary points we need to partially differentiate *f*.

$$f = y^3 - x^3 + 3xy + 1$$

$$\frac{\partial f}{\partial x} = 0 - 3x^2 + 3y = 0 \quad \text{implies that } y = x^2$$

$$\frac{\partial f}{\partial y} = 3y^2 - 0 + 3x = 0 \quad \text{implies that } 3y^2 + 3x = 0$$

Substituting $y = x^2$ into the bottom equation $3y^2 + 3x = 0$ gives

$$3\left(x^2\right)^2 + 3x = 3x^4 + 3x = 3x\left(x^3 + 1\right) = 0$$

From this we have $x = 0,\ x = -1$. Substituting these into $y = x^2$ yields that

$$y = 0,\ y = 1 \quad \text{respectively}$$

Our stationary points are $(0, \quad 0)$ and $(-1, \quad 1)$. We need to decide whether these points are maximum, minimum or saddle points. This means we need to differentiate again:

$$\frac{\partial f}{\partial x} = -3x^2 + 3y$$

$$\frac{\partial^2 f}{\partial x^2} = -6x$$

$$\frac{\partial f}{\partial y} = 3y^2 + 3x$$

$$\frac{\partial^2 f}{\partial y^2} = 6y$$

The mixed partial derivative is given by

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3y^2 + 3x\right) = 3$$

Substituting these $\frac{\partial^2 f}{\partial x^2} = -6x$, $\frac{\partial^2 f}{\partial y^2} = 6y$ and $\frac{\partial^2 f}{\partial x \partial y} = 3$ into formula (15.11):

$$\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial y^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial y}\right)^2 = (-6x)(6y) - 3^2 = -36xy - 9 \qquad (*)$$

Substituting the stationary point $(0, \quad 0)$ which means that $x = 0,\ y = 0$ into (*):

$$\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial y^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial y}\right)^2 = -36(0)(0) - 9 = -9 < 0$$

The stationary point $(0, \quad 0)$ is a saddle point.

Next we test the other stationary point $(-1, \quad 1)$ which means that $x = -1,\ y = 1$. Substituting this into (*) gives

$$\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial y^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial y}\right)^2 = -36(-1)(1) - 9 = 36 - 9 > 0$$

Thus $(-1, \quad 1)$ is a maximum or minimum. Since

$$\frac{\partial^2 f}{\partial x^2} = -6x = -6\times(-1) = 6 > 0$$

therefore $(-1,\quad 1)$ is a minimum.

We have $(0,\quad 0)$ is a saddle point and $(-1,\quad 1)$ is a minimum.

3. We need to find the values of x and y such that $C = 4xy + \frac{72}{y} + \frac{48}{x}$ is a minimum. *How?*

Apply partial differentiation and equate to zero to get the stationary points first:

$$C = 4xy + \frac{72}{y} + \frac{48}{x} = 4xy + 72y^{-1} + 48x^{-1}$$

$$\frac{\partial C}{\partial x} = 4y + 0 - 48x^{-2} = 0 \quad \text{implies} \quad 4y = 48x^{-2} = \frac{48}{x^2}$$

$$\frac{\partial C}{\partial y} = 4x - 72y^{-2} + 0 = 0 \quad \text{implies} \quad 4x = \frac{72}{y^2}$$

Dividing the last two equations by 4 gives

$$y = \frac{12}{x^2}, \qquad x = \frac{18}{y^2}$$

Substituting $y = \frac{12}{x^2}$ into $x = \frac{18}{y^2}$ gives

$$x = \frac{18}{\left(12/x^2\right)^2} = \frac{18x^4}{144}$$

$$144x - 18x^4 = 0$$

$$18x\left(8 - x^3\right) = 0$$

Solving the last equation gives $x = 0,\ x = 2$. We are given that $x \neq 0$ therefore $x = 2$.

Substituting this $x = 2$ into $y = \frac{12}{x^2}$ gives $y = \frac{12}{2^2} = 3$. Thus $(2,\quad 3)$ is a stationary point.

We need to show that this stationary point is indeed a minimum. Determining the second partial derivatives

$$\frac{\partial C}{\partial x} = 4y - 48x^{-2}$$

$$\frac{\partial^2 C}{\partial x^2} = 98x^{-3}$$

$$\frac{\partial C}{\partial y} = 4x - 72y^{-2}$$

$$\frac{\partial^2 C}{\partial y^2} = 144y^{-3}$$

The mixed partial derivative is given by

$$\frac{\partial^2 C}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial C}{\partial y}\right) = \frac{\partial}{\partial x}\left(4x - 72y^{-2}\right) = 4$$

Putting all these $\frac{\partial^2 C}{\partial x^2} = 98x^{-3}$, $\frac{\partial^2 C}{\partial y^2} = 144y^{-3}$ and $\frac{\partial^2 C}{\partial x \partial y} = 4$ into formula (15.11) yields

$$\left(\frac{\partial^2 C}{\partial x^2}\right)\left(\frac{\partial^2 C}{\partial y^2}\right)-\left(\frac{\partial^2 C}{\partial x\partial y}\right)^2=\left(98x^{-3}\right)\left(144y^{-3}\right)-4^2$$

$$=\frac{98\times 144}{x^3y^3}-16$$

Substituting $x=2$ and $y=3$ into the above gives

$$\left(\frac{\partial^2 C}{\partial x^2}\right)\left(\frac{\partial^2 C}{\partial y^2}\right)-\left(\frac{\partial^2 C}{\partial x\partial y}\right)^2=\frac{98\times 144}{2^3 3^3}-16>0$$

We have a maximum or minimum at $x=2$ and $y=3$. Applying

$$\frac{\partial^2 C}{\partial x^2}=98x^{-3}=\frac{98}{x^3}$$

Putting $x=2$ into $\frac{\partial^2 C}{\partial x^2}=\frac{98}{x^3}$ gives a positive answer therefore we have a minimum at $x=2$ and $y=3$.

4. We are given that $k=4\pi^2\frac{m}{T^2}$. Note that k is function of m and T therefore

$$\Delta k\cong\frac{\partial k}{\partial m}\Delta m+\frac{\partial k}{\partial T}\Delta T \qquad (*)$$

Next we determine each component of (*). From $k=4\pi^2\frac{m}{T^2}$ we have

$$k=4\pi^2\frac{m}{T^2}=4\pi^2 mT^{-2}$$

$$\frac{\partial k}{\partial m}=\frac{4\pi^2}{T^2},\qquad \frac{\partial k}{\partial T}=(-2)4\pi^2 mT^{-3}=(-2)4\pi^2\frac{m}{T^3}$$

What is Δm equal to?

$\pm 1.5\%$ of $m=\pm\frac{1.5}{100}m=\pm 0.015m$. This means that $\Delta m=\pm 0.015m$.

What is ΔT equal to?

$\pm 2\%$ of $T=\pm\frac{2}{100}T=\pm 0.02T$. We have $\Delta T=\pm 0.02T$.

Substituting $\frac{\partial k}{\partial m}=\frac{4\pi^2}{T^2}$, $\frac{\partial k}{\partial T}=(-2)4\pi^2\frac{m}{T^3}$, $\Delta m=\pm 0.015m$ and $\Delta T=\pm 0.02T$ into (*) gives

$$\Delta k\cong\frac{\partial k}{\partial m}\Delta m+\frac{\partial k}{\partial T}\Delta T$$

$$=\frac{4\pi^2}{T^2}(\pm 0.015m)+(-2)4\pi^2\frac{m}{T^3}(\pm 0.02T)$$

$$=\frac{4\pi^2 m}{T^2}(\pm 0.015)-\frac{4\pi^2 m}{T^2}(\pm 0.02)$$

$$=\frac{4\pi^2 m}{T^2}\left[(\pm 0.015)-(\pm 0.02)\right]$$

$$=k\left[(\pm 0.015)-(\pm 0.02)\right] \qquad \left[\text{Because } k=\frac{4\pi^2 m}{T^2}\right]$$

What is the maximum error in measuring k?
The maximum error occurs when the signs $\pm$ in the above are different:

$$\Delta k \cong k\left[(+0.015)-(-0.02)\right]$$
$$= k\left[0.035\right]$$

The largest percentage error in the measurement of *k* is 3.5%.

5. (a) We need to partial differentiate the given function $u = 3x^2 + 6x^2y - 4xy^2 - y^3$:

$$u = 3x^2 + 6x^2y - 4xy^2 - y^3$$
$$\frac{\partial u}{\partial x} = 6x + 12xy - 4y^2$$
$$\frac{\partial^2 u}{\partial x^2} = 6 + 12y$$

Similarly we have

$$u = 3x^2 + 6x^2y - 4xy^2 - y^3$$
$$\frac{\partial u}{\partial y} = 6x^2 - 8xy - 3y^2$$
$$\frac{\partial^2 u}{\partial y^2} = -8x - 6y$$

The mixed partial derivative is given by

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[6x^2 - 8xy - 3y^2\right]$$
$$= 12x - 8y$$

(b) We need to show that for the given *u* we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = c^2 \frac{\partial u}{\partial t}$. We have

$$u = \sin(2x)\sin(4y)e^{-5t}$$
$$\frac{\partial u}{\partial x} = 2\cos(2x)\sin(4y)e^{-5t}$$
$$\frac{\partial^2 u}{\partial x^2} = -4\sin(2x)\sin(4y)e^{-5t}$$

Similarly we have

$$u = \sin(2x)\sin(4y)e^{-5t}$$
$$\frac{\partial u}{\partial y} = 4\sin(2x)\cos(4y)e^{-5t}$$
$$\frac{\partial^2 u}{\partial y^2} = -16\sin(2x)\sin(4y)e^{-5t}$$

Additionally we have

$$u = \sin(2x)\sin(4y)e^{-5t}$$
$$\frac{\partial u}{\partial t} = -5\sin(2x)\sin(4y)e^{-5t}$$

Substituting the above $\frac{\partial^2 u}{\partial x^2} = -4\sin(2x)\sin(4y)e^{-5t}$ and $\frac{\partial^2 u}{\partial y^2} = -16\sin(2x)\sin(4y)e^{-5t}$ into the left hand side of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = c^2 \frac{\partial u}{\partial t}$ gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= -4\sin(2x)\sin(4y)e^{-5t} - 16\sin(2x)\sin(4y)e^{-5t}\\ &= -20\sin(2x)\sin(4y)e^{-5t}\\ &= 4\left[-5\sin(2x)\sin(4y)e^{-5t}\right]\\ &= 4\frac{\partial u}{\partial t} \qquad \left[\text{Because } \frac{\partial u}{\partial t} = -5\sin(2x)\sin(4y)e^{-5t}\right]\end{aligned}$$

Hence $c^2 = 4$ which gives $c = 2$.

6. (a) (i) We are given $z = x^3 + 5x^2 y + 2y^3$. We need to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x}$.

$$z = x^3 + 5x^2 y + 2y^3$$
$$\frac{\partial z}{\partial x} = 3x^2 + 10xy$$
$$\frac{\partial^2 z}{\partial x^2} = 6x + 10y$$
$$\frac{\partial z}{\partial y} = 5x^2 + 6y^2$$

The mixed partial derivative can be determined as follows:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 + 10xy\right) = 10x$$

(ii) Similarly for $z = e^x \cos(y)$ we need to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x}$:

$$z = e^x \cos(y)$$
$$\frac{\partial z}{\partial x} = e^x \cos(y)$$
$$\frac{\partial^2 z}{\partial x^2} = e^x \cos(y)$$
$$\frac{\partial z}{\partial y} = -e^x \sin(y)$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[e^x \cos(y)\right] = -e^x \sin(y)$$

(b) We are given $z = e^{px}\left(x\cos(y) - y\sin(y)\right)$. We need to find a value of p which satisfies the following: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

$$z = e^{px}\left(x\cos(y) - y\sin(y)\right)$$

$$\frac{\partial z}{\partial x} \underbrace{=}_{\text{By the Product Rule}} pe^{px}\left(x\cos(y) - y\sin(y)\right) + e^{px}\cos(y)$$

$$= e^{px}\left[px\cos(y) - py\sin(y) + \cos(y)\right]$$

$$\frac{\partial^2 z}{\partial x^2} \underbrace{=}_{\text{By the Product Rule}} pe^{px}\left[px\cos(y) - py\sin(y) + \cos(y)\right] + e^{px}p\cos(y)$$

$$= pe^{px}\left[px\cos(y) - py\sin(y) + 2\cos(y)\right]$$

Similarly we have

$$z = e^{px}\left(x\cos(y) - y\sin(y)\right)$$

$$\frac{\partial z}{\partial y} = e^{px}\left(-x\sin(y) - \underbrace{\left[(1)\sin(y) + y\cos(y)\right]}_{\text{By the Product Rule}}\right)$$

$$= e^{px}\left(-x\sin(y) - \sin(y) - y\cos(y)\right)$$

$$\frac{\partial^2 z}{\partial y^2} = e^{px}\left(-x\cos(y) - \cos(y) - \underbrace{\left[(1)\cos(y) - y\sin(y)\right]}_{\text{By the Product Rule}}\right)$$

$$= e^{px}\left(-x\cos(y) - \cos(y) - \cos(y) + y\sin(y)\right)$$

$$= e^{px}\left(-x\cos(y) - 2\cos(y) + y\sin(y)\right)$$

Adding the two second derivatives

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = pe^{px}\left[px\cos(y) - py\sin(y) + 2\cos(y)\right] + e^{px}\left[-x\cos(y) - 2\cos(y) + y\sin(y)\right]$$

$$= e^{px}\begin{bmatrix} p^2x\cos(y) - x\cos(y) - p^2y\sin(y) + y\sin(y) \\ + 2p\cos(y) - 2\cos(y) \end{bmatrix}$$

$$= e^{px}\left[\left(p^2 - 1\right)x\cos(y) + \left(1 - p^2\right)y\sin(y) + 2(p-1)\cos(y)\right]$$

Clearly this is zero when $p = 1$ because **all** the terms in the square bracket are zero at $p = 1$.

(c) We need to find the stationary points of $f(x, \quad y) = 2x^3 + 6xy^2 - 3y^3 - 150x$:

$$f = 2x^3 + 6xy^2 - 3y^3 - 150x$$

$$\frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150 = 0 \quad \Rightarrow \quad x^2 + y^2 = \frac{150}{6} = 25$$

$$\frac{\partial f}{\partial y} = 12xy - 9y^2 = 0 \quad \Rightarrow \quad 3y(4x - 3y) = 0$$

From the bottom equation we have $y = 0$ or $4x - 3y = 0 \quad \Rightarrow \quad x = \frac{3y}{4}$.

Substituting $y = 0$ into the first equation $x^2 + y^2 = 25$ gives $x^2 = 25 \Rightarrow \quad x = \pm 5$.

Substituting $x = \frac{3y}{4}$ into the first equation $x^2 + y^2 = 25$ gives

$$x^2+y^2=\left(\frac{3y}{4}\right)^2+y^2$$

$$=\frac{9y^2}{16}+y^2=\frac{9y^2+16y^2}{16}=\frac{25y^2}{16}=25$$

$$y^2=16 \text{ gives } y=\pm 4$$

If $y=4$ then $x=\frac{3y}{4}=\frac{3\times 4}{4}=3$. Similarly if $y=-4$ then $x=-3$.

We have four stationary points $(5,\ \ 0)$, $(-5,\ \ 0)$, $(3,\ \ 4)$ and $(-3,\ \ -4)$. We need to check the nature of each of these points. This means we need to find the second partial derivatives:

$$f=2x^3+6xy^2-3y^3-150x$$

$$\frac{\partial f}{\partial x}=6x^2+6y^2-150$$

$$\frac{\partial^2 f}{\partial x^2}=12x$$

$$\frac{\partial f}{\partial y}=12xy-9y^2$$

$$\frac{\partial^2 f}{\partial y^2}=12x-18y$$

We also need to determine the mixed partial derivative

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$=\frac{\partial}{\partial x}\left(12xy-9y^2\right)=12y$$

Substituting $\frac{\partial^2 f}{\partial x^2}=12x$, $\frac{\partial^2 f}{\partial y^2}=12x-18y$ and $\frac{\partial^2 f}{\partial x\partial y}=12y$ into formula (15.11) gives

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2=12x\left(12x-18y\right)-\left(12y\right)^2$$

$$=144x^2-216xy-144y^2 \qquad (\dagger)$$

We test each of the above four stationary points $(5,\ \ 0)$, $(-5,\ \ 0)$, $(3,\ \ 4)$ and $(-3,\ \ -4)$ by substituting these x and y values into (†). At $(5,\ \ 0)$ we have

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2=144x^2-216xy-144y^2$$

$$=\left(144\times 5^2\right)-0-0>0$$

Thus $(5,\ \ 0)$ is a maximum or minimum. Since $\frac{\partial^2 f}{\partial x^2}=12x=12\times 5=60>0$ therefore we have minimum at $(5,\ \ 0)$.

Similarly for $(-5,\ \ 0)$ we have maximum or minimum and $\frac{\partial^2 f}{\partial x^2}=12x=12\times(-5)=-60<0$ therefore this is a maximum.

Testing the point $(3,\quad 4)$ by substituting $x=3,\ \ y=4$ into (†):

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 144x^2-216xy-144y^2$$

$$=\left(144\times 3^2\right)-\left(216\times 3\times 4\right)-\left(144\times 4^2\right)$$

$$=144\left(9-16\right)-\left(216\times 3\times 4\right)<0$$

Thus $(3,\quad 4)$ is a saddle point. The last stationary point $(-3,\quad -4)$ is also a saddle point because substituting $x=-3,\ \ y=-4$ into (†) gives the same result as above.