Complete solutions to Exercise 11

1. (i) $\begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}$ (ii) $\begin{pmatrix} -42 & 0 \\ 0 & -42 \end{pmatrix}$ (iii) $\begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$

2. Note that the values $a,\ b,\ c$ and d satisfy the inverse matrix. By using (11.4) we have

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{(1\times 4)-(3\times 2)}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Thus $a=-2,\ b=1,\ c=3/2$ and $d=-1/2$.

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3. $\mathbf{A}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ and $\mathbf{A}^{10} = \begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$

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4. (i) Multiplying out the matrix yields

$$\left.\begin{aligned} 5x+6y&=1 \\ 4x+5y&=0 \end{aligned}\right\} \text{ which gives } x=5,\ \ y=-4$$

(ii) Similarly we have

$$\left.\begin{aligned} 5a+6b&=0 \\ 4a+5b&=1 \end{aligned}\right\} \text{ which gives } a=-6,\ \ b=5$$

(iii) Hence $\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -6 \\ -4 & 5 \end{pmatrix}$

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5. (a) We have $\mathbf{A}^2 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$. Substituting this into

$$\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} - 5\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 10 & 5 \\ 0 & 15 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 4-10+6 & 5-5+0 \\ 0-0+0 & 9-15+6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

(b) We have $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix}$. Thus

$$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 12 & 3 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

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6. We have

$$\begin{pmatrix} 5 & -1 & -2 \\ 10 & -2 & -4 \\ 15 & -3 & -6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 2 & 3 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

It is possible to multiply 2 non zero matrices to give the zero matrix. However this is not true for numbers.

(11.4) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

7. Expand along the bottom row which are all zeros, thus determinant is zero.

$$\det\begin{pmatrix}1 & 15 & -6 & 7 & 77\\ 0 & 2 & -973 & 533 & 207\\ 0 & 0 & 3 & -66 & 1057\\ 0 & 0 & 0 & 4 & 855\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}=0$$

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8. Writing the given equations in matrix form

$$\begin{pmatrix}100 & -30 & -40\\ -30 & 140 & -60\\ -40 & -60 & 170\end{pmatrix}\begin{pmatrix}I_1\\ I_2\\ I_3\end{pmatrix}=\begin{pmatrix}-8\\ -2\\ 10\end{pmatrix}$$

Putting this in an augmented matrix gives

$$\begin{matrix}\text{Row 1}\\ \text{Row 2}\\ \text{Row 3}\end{matrix}\left(\begin{array}{ccc|c}100 & -30 & -40 & -8\\ -30 & 140 & -60 & -2\\ -40 & -60 & 170 & 10\end{array}\right)$$

Divide the first row by 100 and the other two rows by 10:

$$\begin{matrix}R_1\\ R_2\\ R_3\end{matrix}\left(\begin{array}{ccc|c}1 & -0.3 & -0.4 & -0.08\\ -3 & 14 & -6 & -0.2\\ -4 & -6 & 17 & 1\end{array}\right)$$

$$\begin{matrix}R_1\\ R'_2=R_2+3R_1\\ R'_3=R_3+4R_1\end{matrix}\left(\begin{array}{ccc|c}1 & -0.3 & -0.4 & -0.08\\ 0 & 13.1 & -7.2 & -0.44\\ 0 & -7.2 & 15.4 & 0.68\end{array}\right)$$

$$\begin{matrix}R_1\\ R'_2\\ R''_3=R'_3+\dfrac{7.2}{13.1}R'_2\end{matrix}\left(\begin{array}{ccc|c}1 & -0.3 & -0.4 & -0.08\\ 0 & 13.1 & -7.2 & -0.44\\ 0 & 0 & 11.443 & 0.43817\end{array}\right)$$

From the last row 3, R''_3, we have

$$11.443I_3=0.43817 \text{ which gives } I_3=\frac{0.43817}{11.443}=38.292\times10^{-3}$$

By substituting $I_3=38.292\times10^{-3}$ into R'_2 we have

$$13.1I_2+\left(-7.2\times38.292\times10^{-3}\right)=-0.44 \text{ gives } I_2=-12.542\times10^{-3}$$

By substituting the above into R_1 we have

$$I_1-\left[0.3\times\left(-12.542\times10^{-3}\right)\right]-\left(0.4\times38.292\times10^{-3}\right)=-0.08$$

$$I_1=-68.446\times10^{-3}$$

We have $I_1=-68.45\text{mA}$, $I_2=-12.54\text{mA}$ and $I_3=38.29\text{mA}$

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9. Writing the given equations in matrix form we have

$$\begin{pmatrix}9 & -3 & -5\\ -3 & 16.5 & -0.5\\ -5 & -0.5 & 20.5\end{pmatrix}\begin{pmatrix}i_1\\ i_2\\ i_3\end{pmatrix}=\begin{pmatrix}12\times10^{-3}\\ 0\\ 15\times10^{-3}\end{pmatrix}\qquad(*)$$

Let **A** be the 3×3 matrix in (*). We first find the determinant of **A**.

$$\det\begin{pmatrix} 9 & -3 & -5 \\ -3 & 16.5 & -0.5 \\ -5 & -0.5 & 20.5 \end{pmatrix} = 9\det\begin{pmatrix} 16.5 & -0.5 \\ -0.5 & 20.5 \end{pmatrix} + 3\det\begin{pmatrix} -3 & -0.5 \\ -5 & 20.5 \end{pmatrix} - 5\det\begin{pmatrix} -3 & 16.5 \\ -5 & -0.5 \end{pmatrix}$$

$$= (9\times 338) + (3\times(-64)) - (5\times 84) = 2430$$

To obtain the inverse matrix we have to evaluate the cofactors and then transpose these to give the $adj\mathbf{A}$ and to find the inverse we use

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\,adj\mathbf{A}$$

Thus in our case we have

$$\mathbf{A}^{-1} = \frac{1}{2430}\begin{pmatrix} 338 & 64 & 84 \\ 64 & 159.5 & 19.5 \\ 84 & 19.5 & 139.5 \end{pmatrix}$$

To find the values of i_1, i_2 and i_3 we multiply $\mathbf{A}^{-1}$ and the right hand side of (*):

$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \frac{1}{2430}\begin{pmatrix} 338 & 64 & 84 \\ 64 & 159.5 & 19.5 \\ 84 & 19.5 & 139.5 \end{pmatrix}\begin{pmatrix} 12\times 10^{-3} \\ 0 \\ 15\times 10^{-3} \end{pmatrix} = \frac{1}{2430}\begin{pmatrix} 5.316 \\ 1.0605 \\ 3.1005 \end{pmatrix}$$

$$= \begin{pmatrix} 5.316/2430 \\ 1.0605/2430 \\ 3.1005/2430 \end{pmatrix} = \begin{pmatrix} 2.187\times 10^{-3} \\ 0.436\times 10^{-3} \\ 1.276\times 10^{-3} \end{pmatrix}$$

Remember 10^{-3} is milli so we have

$$i_1 = 2.19\,\text{mA},\ i_2 = 0.44\,\text{mA} \text{ and } i_3 = 1.28\,\text{mA}$$

Since $\det\mathbf{A} = 2430 \neq 0$, we have a unique solution.

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10. To find the eigenvalues, λ, we subtract these λ from the leading diagonal and evaluate the resulting determinant:

$$\det\begin{pmatrix} \frac{3k}{2m}-\lambda & \frac{k}{2m} \\ \frac{k}{m} & \frac{k}{m}-\lambda \end{pmatrix} = \left(\frac{3k}{2m}-\lambda\right)\left(\frac{k}{m}-\lambda\right) - \frac{k}{m}\frac{k}{2m}$$

$$= \left(\frac{3k^2}{2m^2} - \frac{3k\lambda}{2m} - \frac{k\lambda}{m} + \lambda^2\right) - \frac{k^2}{2m^2} = \lambda^2 - \frac{5k\lambda}{2m} + \frac{k^2}{m^2}$$

Remember to find the eigenvalues we equate the determinant to zero.

$$\lambda^2 - \frac{5\lambda k}{2m} + \frac{k^2}{m^2} = \left(\lambda - \frac{k}{2m}\right)\left(\lambda - \frac{2k}{m}\right) = 0$$

We have $\lambda_1 = \dfrac{k}{2m}$ and $\lambda_2 = \dfrac{2k}{m}$. Substituting these into T gives

$$T_1 = \frac{2\pi}{\sqrt{-k/2m}} = \frac{2\pi\sqrt{2m}}{\sqrt{-k}} = 2\pi\sqrt{-\frac{2m}{k}},\quad T_2 = \frac{2\pi}{\sqrt{-2k/m}} = \frac{2\pi\sqrt{m}}{\sqrt{2}\sqrt{-k}} = \pi\frac{\sqrt{2}\sqrt{m}}{\sqrt{-k}} = \pi\sqrt{-\frac{2m}{k}}$$

11. We have

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ -1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similarly we have

$$\mathbf{y} = \begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{Cx} \text{ gives } \mathbf{C} = \begin{pmatrix} 1 & 4 \end{pmatrix}$$

(i) To find the eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 2-\lambda & 2 \\ -1 & 5-\lambda \end{pmatrix}$$

$$= (2-\lambda)(5-\lambda) + 2 = 10 - 7\lambda + \lambda^2 + 2 = \lambda^2 - 7\lambda + 12$$

Equating the final quadratic to zero gives

$$\lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3) = 0$$

$$\lambda_1 = 4,\ \lambda_2 = 3$$

(ii) Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigenvector for $\lambda_1 = 4$. Then

$$\begin{pmatrix} 2-4 & 2 \\ -1 & 5-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out the matrices gives

$$-2x + 2y = 0$$

$$-x + y = 0$$

Thus we have $x = a$ and $y = a$ where a is non zero, $\mathbf{v} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigenvector for $\lambda_2 = 3$:

$$\begin{pmatrix} 2-3 & 2 \\ -1 & 5-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $\mathbf{u} = a\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the eigenvector for $\lambda_2 = 3$.

12. We have

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} s & 1 \\ -3 & s+2 \end{pmatrix}$$

The inverse of this is given by

$$\begin{pmatrix} s & 1 \\ -3 & s+2 \end{pmatrix}^{-1} = \frac{1}{s(s+2)+3}\begin{pmatrix} s+2 & -1 \\ 3 & s \end{pmatrix} = \frac{1}{s^2+2s+3}\begin{pmatrix} s+2 & -1 \\ 3 & s \end{pmatrix}$$

$$T.F. = \frac{1}{s^2+2s+3}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} s+2 & -1 \\ 3 & s \end{pmatrix}\begin{pmatrix} 1 \\ 0.6 \end{pmatrix}$$

$$= \frac{1}{s^2+2s+3}\begin{pmatrix} s+2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.6 \end{pmatrix} = \frac{1}{s^2+2s+3}(s+1.4) = \frac{s+1.4}{s^2+2s+3}$$

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13. We have

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\mathbf{C}_T^{-1}p(\mathbf{A})$$

where $\mathbf{C}_T = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{pmatrix}$ and $p(\mathbf{A}) = \mathbf{A}^3 + 9\mathbf{A}^2 - 3\mathbf{A} + 5\mathbf{I}$. Using MAPLE:

```
> A:=matrix([[-5, 2, 1], [-6, 5, 9], [7, -1, -4]]);
```

$$A := \begin{bmatrix} -5 & 2 & 1 \\ -6 & 5 & 9 \\ 7 & -1 & -4 \end{bmatrix}$$

```
> B:=matrix([[-1], [3], [8]]);
```

$$B := \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

```
> AB:=evalm(A&*B);
```

$$AB := \begin{bmatrix} 19 \\ 93 \\ -42 \end{bmatrix}$$

```
> A2B:=evalm(A^2&*B);
```

$$A2B := \begin{bmatrix} 49 \\ -27 \\ 208 \end{bmatrix}$$

```
> C:=matrix([[-1, 19, 49], [3, 93, -27], [8, -42, 208]]);
```

$$C := \begin{bmatrix} -1 & 19 & 49 \\ 3 & 93 & -27 \\ 8 & -42 & 208 \end{bmatrix}$$

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and
unprotected
```

```
> invC:=inverse(C);
```

$$invC := \begin{bmatrix} \frac{-607}{2560} & \frac{601}{7680} & \frac{169}{2560} \\ \frac{7}{640} & \frac{1}{128} & \frac{-1}{640} \\ \frac{29}{2560} & \frac{-11}{7680} & \frac{1}{512} \end{bmatrix}$$

```
> id:=evalm(array(identity,1..3,1..3));
```

$$id := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> p(A):=evalm(A^3+9*A^2-3*A+5*id);
```

$$\mathrm{p}(A) := \begin{bmatrix} 169 & 11 & 53 \\ 267 & 169 & 87 \\ -229 & 57 & 147 \end{bmatrix}$$

```
> G:=matrix([[0, 0, 1]]);
```

$$G := [0 \quad 0 \quad 1]$$

```
> F:=evalm(G&*invC&*p(A));
```

$$F := \begin{bmatrix} \frac{2777}{2560} & \frac{-47}{7680} & \frac{1953}{2560} \end{bmatrix}$$

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14. (a)

```
> A:=matrix([[6, 3, -11], [-2, 15, 3], [7, 9, 12]]);
```

$$A := \begin{bmatrix} 6 & 3 & -11 \\ -2 & 15 & 3 \\ 7 & 9 & 12 \end{bmatrix}$$

```
> ID:=matrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]]);
```

$$ID := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> p(A):=A^3+5*A^2-7*A+6*ID;
```

$$\mathrm{p}(A) := A^3 + 5\,A^2 - 7\,A + 6\,ID$$

```
> evalm(%);
```

$$\begin{bmatrix} -1804 & -2583 & -2727 \\ 12 & 5685 & 2699 \\ 1269 & 6387 & 1124 \end{bmatrix}$$

(b)

```
> B:=matrix([[9.31, -6.11, 2.3], [-2.75, 3.93, -6.51], [-
8.99, 12.23, -1.75]]);
```

$$B := \begin{bmatrix} 9.31 & -6.11 & 2.3 \\ -2.75 & 3.93 & -6.51 \\ -8.99 & 12.23 & -1.75 \end{bmatrix}$$

```
> p(B):=B^3-6.73*B^2+5.29*B+6*ID;
```

$$\mathrm{p}(B) := B^3 - 6.73\, B^2 + 5.29\, B + 6\, ID$$

```
> evalm(%);
```

$$\begin{bmatrix} -99.916881 & 308.625987 & 61.374886 \\ 357.222199 & -226.617083 & 498.786883 \\ 340.063455 & -732.143995 & 56.442441 \end{bmatrix}$$

15.

```
> A:=matrix([[2.2, 3.1, 6.7, -2.7], [1.1, -2.8, 5.6, -
5.5], [3.4, 2.7, -1.9, -4.6], [-7.6, 8.2, 1.5, 4.2]]);
```

$$A := \begin{bmatrix} 2.2 & 3.1 & 6.7 & -2.7 \\ 1.1 & -2.8 & 5.6 & -5.5 \\ 3.4 & 2.7 & -1.9 & -4.6 \\ -7.6 & 8.2 & 1.5 & 4.2 \end{bmatrix}$$

```
> B:=matrix([[-1.1], [2.3], [1.7], [-4.1]]);
```

$$B := \begin{bmatrix} -1.1 \\ 2.3 \\ 1.7 \\ -4.1 \end{bmatrix}$$

```
> C:=matrix([[9.1, 6.2, 4.7, 5.5]]);
```

$$C := \begin{bmatrix} 9.1 & 6.2 & 4.7 & 5.5 \end{bmatrix}$$

```
> id:=evalm(array(identity,1..4,1..4)):
> simplify(evalm(inverse(s*id-A))):
> simplify(evalm(C&*(%)&*B));
```

$$\left[-0.2000000000 \frac{-0.101376950\ 10^9\, s - 0.28513650\ 10^8\, s^2 + 515500.\, s^3 + 0.768846877\ 10^9}{10000.\, s^4 - 17000.\, s^3 - 253500.\, s^2 - 0.1310550\ 10^7\, s - 0.24402843\ 10^8} \right]$$
