Complete solutions to Exercise 15(c)

1. (a)

$$f(x,y) = xy + \frac{1}{2}y^2 - 7y - 4x$$

$$\frac{\partial f}{\partial x} = y - 4, \qquad \frac{\partial f}{\partial y} = x + y - 7$$

We have the simultaneous equations

$$y - 4 = 0 \text{ gives } y = 4$$

$$x + y - 7 = 0$$

Substituting $y = 4$ gives $x = 3$. Hence at (3,4) we have a stationary point.

(b) We have

$$f(x,y) = x^2 + y^2 + 4xy - 5x - 4y$$

$$\frac{\partial f}{\partial x} = 2x + 4y - 5, \quad \frac{\partial f}{\partial y} = 2y + 4x - 4$$

We need to solve the simultaneous equations

$$2x + 4y - 5 = 0$$

$$2y + 4x - 4 = 0$$

Solving these gives $x = 0.5$, $y = 1$. At (0.5,1) we have a stationary point.

(c) We have

$$f(x,y) = x^3 + y^2 + xy + 22y$$

$$\frac{\partial f}{\partial x} = 3x^2 + y, \quad \frac{\partial f}{\partial y} = 2y + x + 22$$

We need to solve the simultaneous equations

$$3x^2 + y = 0$$

$$2y + x + 22 = 0$$

From the first equation we have $y = -3x^2$.

Substituting $y = -3x^2$ into the second equation.

$$2(-3x^2) + x + 22 = 0$$

$$-6x^2 + x + 22 = 0$$

$$6x^2 - x - 22 = 0$$

$$(6x + 11)(x - 2) = 0 \text{ which gives } x = -11/6,\ x = 2$$

Substituting $x = 2$ into $y = -3x^2$ gives $y = -3(2)^2 = -12$

At (2,-12) we have a stationary point.

For $x = -11/6$;

$$y = -3\left(\frac{-11}{6}\right)^2 = -3 \times \frac{121}{36} = -\frac{121}{12}$$

At $\left(-\frac{11}{6}, -\frac{121}{12}\right)$ we also have a stationary point.

2. (a) We are given

$$f(x,y) = x^2 + y^2 + 6xy - 10x - 14y$$

$$\frac{\partial f}{\partial x} = 2x + 6y - 10, \qquad \frac{\partial f}{\partial y} = 2y + 6x - 14$$

Solving the simultaneous equations

$$2x + 6y - 10 = 0$$

$$2y + 6x - 14 = 0$$

gives $x = 2$ and $y = 1$. *At (2,1) do we have maximum, minimum or a saddle point?* Need to use the second partial derivative test

$$\frac{\partial^2 f}{\partial x^2} = 2, \ \frac{\partial^2 f}{\partial y^2} = 2 \ \text{ and } \ \frac{\partial}{\partial x}(2y + 6x - 14) = 6$$

Substituting these into $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ gives

$$2(2) - 6^2 < 0$$

By (15.13) at (2,1) we have a saddle point.

(b) Let $f = f(x, y)$ then

$$f = x^3 + y^2 - x + y$$

$$\frac{\partial f}{\partial x} = 3x^2 - 1, \quad \frac{\partial f}{\partial y} = 2y + 1$$

Putting both these to zero gives

$$3x^2 - 1 = 0, \ x^2 = \frac{1}{3} \ \text{ which gives } \ x = \pm\frac{1}{\sqrt{3}}$$

Also

$$2y + 1 = 0 \ \text{ gives } \ y = -\frac{1}{2}$$

We have stationary points at $\left(\frac{1}{\sqrt{3}}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{2}\right)$. We use the second partial derivative test to find the nature of each stationary point.

$$\frac{\partial^2 f}{\partial x^2} = 6x, \ \frac{\partial^2 f}{\partial y^2} = 2 \ \text{ and } \ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2y + 1) = 0$$

Substituting these into $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ yields

$$6x(2) - 0^2 = 12x$$

Putting $x = \frac{1}{\sqrt{3}}$, $y = -\frac{1}{2}$ into $12x$

$$12\left(\frac{1}{\sqrt{3}}\right) > 0 \ \text{ and } \ \frac{\partial^2 f}{\partial x^2} = 6x = 6 \times \frac{1}{\sqrt{3}} > 0$$

(15.13) $$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right) < 0$$ (Saddle)

By (15.11), at $\left(\frac{1}{\sqrt{3}},\ -\frac{1}{2}\right)$ we have a minimum. Next we test $\left(-\frac{1}{\sqrt{3}},\ -\frac{1}{2}\right)$.

Substituting $x=-\frac{1}{\sqrt{3}}$ and $y=-\frac{1}{2}$ into $12x$ gives $12\left(-\frac{1}{\sqrt{3}}\right)<0$. By (15.13), at $\left(-\frac{1}{\sqrt{3}},\ -\frac{1}{2}\right)$ we have a saddle point.

(c) Let $f=f(x,\ y)$ then

$$f=x^2+y^3+4xy-11x-18y$$

$$\frac{\partial f}{\partial x}=2x+4y-11,\qquad \frac{\partial f}{\partial y}=3y^2+4x-18$$

Putting the partial derivatives to zero

$$2x+4y-11=0$$

$$3y^2+4x-18=0$$

From the first equation we have

$$x=\frac{11-4y}{2}$$

Substituting this into the second equation gives

$$3y^2+4\left(\frac{11-4y}{2}\right)-18=3y^2+2(11-4y)-18$$

$$=3y^2+22-8y-18$$

Hence

$$3y^2-8y+4=0$$

$$(y-2)(3y-2)=0$$

$$y=2,\ \ y=2/3$$

To find x we substitute y values into $x=\frac{11-4y}{2}$

$$y=2,\ x=\frac{11-4(2)}{2}=\frac{3}{2}\ \text{and}\ y=\frac{2}{3},\quad x=\frac{11-4(2/3)}{2}=\frac{25}{6}$$

Hence at $\left(\frac{3}{2},\ 2\right)$ and $\left(\frac{25}{6},\frac{2}{3}\right)$ we have stationary points. Since

$$\frac{\partial f}{\partial x}=2x+4y-11\qquad \frac{\partial f}{\partial y}=3y^2+4x-18$$

$$\frac{\partial^2 f}{\partial x^2}=2\qquad \frac{\partial^2 f}{\partial y^2}=6y$$

Also

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(3y^2+4x-18\right)=4$$

(15.11) $$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x\partial y}\right)>0,\quad \frac{\partial^2 f}{\partial x^2}>0\quad \text{(Minimum)}$$

We have

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2=(2)(6y)-4^2=12y-16$$

Substituting $x=3/2$ and $y=2$ into $12y-16$ gives

$$(12\times 2)-16>0$$

Also $\frac{\partial^2 f}{\partial x^2}=2>0$. At $\left(\frac{3}{2},\ 2\right)$ we have a minimum.

Similarly substituting $x=25/6$ and $y=2/3$ into $12y-16$ gives

$$12\left(\frac{2}{3}\right)-16<0$$

At $\left(\frac{25}{6},\ \frac{2}{3}\right)$ we have a saddle point.

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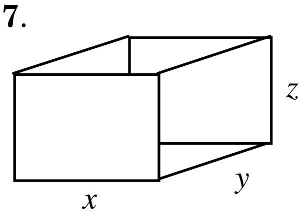
3. Similar to **EXAMPLE 17**.



The volume $xyz=0.9$ and transposing gives

$$z=\frac{0.9}{xy} \qquad (*)$$

Surface area, A, is given by

$$A=xy+2xz+2yz$$
$$=xy+2x\left(\frac{0.9}{xy}\right)+2y\left(\frac{0.9}{xy}\right)$$
$$A=xy+1.8y^{-1}+1.8x^{-1}$$

We first find where the stationary points occur

$$\frac{\partial A}{\partial x}=y-1.8x^{-2},\ \frac{\partial A}{\partial y}=x-1.8y^{-2}$$

Equating these partial derivatives to zero.

$$y-1.8x^{-2}=0,\ y=\frac{1.8}{x^2}\ \text{ which gives }\ x^2y=1.8$$

$$x-1.8y^{-2}=0,\ x=\frac{1.8}{y^2}\ \text{ which gives }\ xy^2=1.8, \qquad \frac{\partial^2 f}{\partial x^2}>0$$

Equating these two equations gives

$$x^2y=xy^2$$

Dividing through by xy

$$x=y$$

Hence

$$xy^2=xx^2=x^3=1.8\ \text{ which gives }\ x=\sqrt[3]{1.8}=1.216$$

Also $y=1.216$

To show that these x and y values produce minimum surface area we need to find the second partial derivatives:

$$\frac{\partial A}{\partial x} = y - 1.8x^{-2}, \qquad \frac{\partial A}{\partial y} = x - 1.8y^{-2}$$

$$\frac{\partial^2 A}{\partial x^2} = 3.6x^{-3}, \qquad \frac{\partial^2 A}{\partial y^2} = 3.6y^{-3}$$

$$\frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - 1.8y^{-2}\right) = 1$$

We have

$$\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial y^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial y}\right)^2 = \left(3.6x^{-3}\right)\left(3.6y^{-3}\right) - 1^2$$

Substituting $x = 1.216$ and $y = 1.216$ into the right hand side gives

$$\left(3.6 \times 1.216^{-3}\right)\left(3.6 \times 1.216^{-3}\right) - 1^2 = 3 > 0$$

Also $\frac{\partial^2 A}{\partial x^2} > 0$. By (15.11), $x = 1.216m$, $y = 1.216m$ gives minimum surface area. From (*)

$$z = \frac{0.9}{xy} = \frac{0.9}{1.216 \times 1.216} = 0.609m$$

The rectangular tank dimensions are $1.216 \times 1.216 \times 0.609$.
The x and y dimensions are approximately twice the z dimension.
Perhaps if we had taken more decimal places we could establish that

$$x = y = 2z$$

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4. Let x, y and z be dimensions of the tank.
Then $xyz = V$. Transposing to make z the subject

$$z = \frac{V}{xy} \qquad (*)$$

The total surface area $A = xy + 2xz + 2yz$

Substituting $z = \frac{V}{xy}$ into A gives

$$A = xy + 2x\left(\frac{V}{xy}\right) + 2y\left(\frac{V}{xy}\right) = xy + \frac{2V}{y} + \frac{2V}{x}$$

$$A = xy + 2Vy^{-1} + 2Vx^{-1}$$

For stationary points;

$$\frac{\partial A}{\partial x} = y - 2Vx^{-2} \quad \text{and} \quad \frac{\partial A}{\partial y} = x - 2Vy^{-2}$$

Equating these to zero and rearranging gives

$$x^2 y = 2V \quad \text{and} \quad xy^2 = 2V$$

Hence $x^2 y = xy^2$. Dividing through by xy gives

$$x = y$$

(15.11) $$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right) > 0, \quad \frac{\partial^2 f}{\partial x^2} > 0 \quad \text{(Minimum)}$$

Substituting $x = y$ into $y = \frac{2V}{x^2}$ gives

$$x = \frac{2V}{x^2}$$
$$x^3 = 2V$$
$$x = (2V)^{1/3}$$

Hence $y = (2V)^{1/3}$ because $x = y$. Need to check minimum.

$$\frac{\partial A}{\partial x} = y - 2Vx^{-2} \qquad \frac{\partial A}{\partial y} = x - 2Vy^{-2}$$

$$\frac{\partial^2 A}{\partial x^2} = 4Vx^{-3} = \frac{4V}{x^3} \qquad \frac{\partial^2 A}{\partial y^2} = 4Vy^{-3} = \frac{4V}{y^3}$$

$$\frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x - 2Vy^{-2}\right) = 1$$

Substituting these into $\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial y^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial y}\right)^2$ gives

$$\left(\frac{4V}{x^3}\right)\left(\frac{4V}{y^3}\right) - 1^2$$

Substituting $x = y = (2V)^{1/3}$ gives

$$\left(\frac{4V}{2V}\right)\left(\frac{4V}{2V}\right) - 1 = 3 > 0$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{4V}{2V} = 2 > 0$$

Hence $x = (2V)^{1/3}$ and $y = (2V)^{1/3}$ gives minimum surface area.

How do we find z? Use (*)

$$z = \frac{V}{xy} = \frac{V}{(2V)^{1/3}(2V)^{1/3}} = \frac{V}{2^{2/3}V^{2/3}} = \frac{V^{1/3}}{2^{2/3}} = \left(\frac{V}{2^2}\right)^{1/3} = \left(\frac{V}{4}\right)^{1/3}$$

Hence the dimensions of the tank are

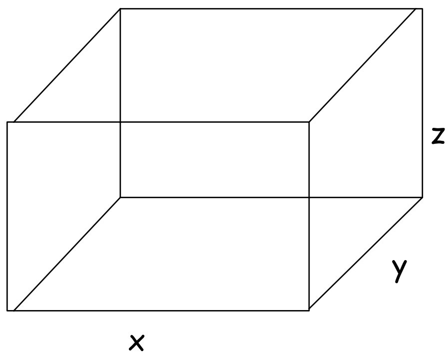
$$x = (2V)^{1/3}, \quad y = (2V)^{1/3} \quad \text{and} \quad z = \left(\frac{V}{4}\right)^{1/3}$$

Note that

$$\frac{1}{2}(2)^{1/3} = \left(\frac{2}{2^3}\right)^{1/3} = \left(\frac{1}{2^2}\right)^{1/3} = \left(\frac{1}{4}\right)^{1/3}$$

Worth noting that $z = \frac{1}{2}x = \frac{1}{2}y$, so that the conjecture at the end of solution 3 is proved.

5. (i) The volume $xyz=9$ where x, y and z are as shown below.



We have $z=\dfrac{9}{xy}$. The total surface area

$$A = xy + 2xz + 2yz$$
$$= xy + 2x\left(\frac{9}{xy}\right) + 2y\left(\frac{9}{xy}\right)$$
$$A = xy + \frac{18}{y} + \frac{18}{x}$$

(ii) For minimum surface area use the formula established in solution 4.

$$x = (2V)^{1/3},\ \ y = (2V)^{1/3} \ \text{ and } \ z = \left(\frac{V}{4}\right)^{1/3}$$

Substituting $V = 9$

$$x = (18)^{1/3} = 2.62 \text{ m},\ \ y = 2.62 \text{ m} \ \text{ and } \ z = \left(\frac{9}{4}\right)^{1/3} = 1.31 \text{ m}$$

Dimensions are correct to 2 d.p.

6. (i) $$T = x^2 + y^2 - x - y + 100 \qquad (\dagger)$$

For stationary points $\dfrac{\partial T}{\partial x} = 2x - 1,\quad \dfrac{\partial T}{\partial y} = 2y - 1$

Equating these to zero gives $x = 1/2$ and $y = 1/2$.
Does the point $(1/2, 1/2)$ in the circular plate give minimum temperature?

$$\frac{\partial^2 T}{\partial x^2} = 2,\quad \frac{\partial^2 T}{\partial y^2} = 2,\quad \frac{\partial^2 T}{\partial x \partial y} = \frac{\partial}{\partial x}(2y - 1) = 0$$

Substituting these into $\left(\dfrac{\partial^2 T}{\partial x^2}\right)\left(\dfrac{\partial^2 T}{\partial y^2}\right) - \left(\dfrac{\partial^2 T}{\partial x \partial y}\right)^2$ gives $2(2) - 0^2 = 4 > 0$

Since $\dfrac{\partial^2 T}{\partial x^2} = 2 > 0$, by (15.11) the values $x = 1/2$ and $y = 1/2$ gives minimum temperature.
To find T at this point, substitute $x = \dfrac{1}{2}$, $y = \dfrac{1}{2}$ into $(\dagger)$

(15.11) $$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right) > 0,\quad \frac{\partial^2 f}{\partial x^2} > 0 \quad \text{(Minimum)}$$

$$T=\frac{1}{4}+\frac{1}{4}-\frac{1}{2}-\frac{1}{2}+100=99.5$$

(ii) We have

$$x^2+y^2=1$$

$$y^2=1-x^2 \text{ which gives } y=\sqrt{1-x^2}$$

Substituting $y=\sqrt{1-x^2}$ and $y^2=1-x^2$ into $T=x^2+y^2-x-y+100$ gives

$$T=x^2+\left(1-x^2\right)-x-\sqrt{1-x^2}+100$$

$$=1-x-\left(1-x^2\right)^{1/2}+100$$

For stationary points: $\frac{dT}{dx}=-1-\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)=-1+\frac{x}{\left(1-x^2\right)^{1/2}}$

Equating this to zero gives

$$\frac{-\left(1-x^2\right)^{1/2}+x}{\left(1-x^2\right)^{1/2}}=0$$

$$x-\left(1-x^2\right)^{1/2}=0$$

$$x^2=1-x^2$$

$$2x^2=1 \text{ gives } x=\pm\frac{1}{\sqrt{2}}$$

To find y we substitute $x=\pm\frac{1}{\sqrt{2}}$ into $y=\sqrt{1-x^2}$

$$y=\pm\sqrt{1-\frac{1}{2}}=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}$$

Stationary points occur at

$$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

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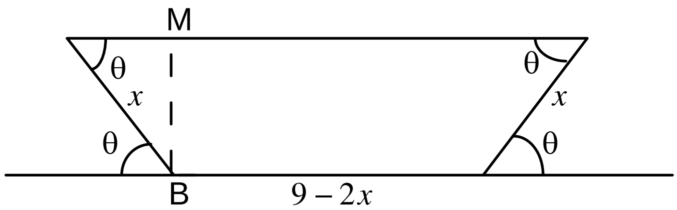
7. We have



The sketch shows the channel in question.

$$\text{Height } BM=x\sin(\theta)$$

$$\text{Rectangular area}=(9-2x)x\sin(\theta)$$

$$\text{Area of both triangles}=x\cos(\theta)x\sin(\theta)=x^2\sin(\theta)\cos(\theta)$$

$$\text{Total area } A=(9-2x)x\sin(\theta)+x^2\sin(\theta)\cos(\theta)$$

For stationary points we need $\frac{\partial A}{\partial x}=0$ and $\frac{\partial A}{\partial \theta}=0$

$$\frac{\partial A}{\partial x}=(9-4x)\sin(\theta)+2x\sin(\theta)\cos(\theta)$$

$$\left[9-4x+2x\cos(\theta)\right]\sin(\theta)=0$$

$$\sin(\theta)=0 \text{ gives } \theta=0$$

This, $\theta=0$, is impossible because if $\theta=0$ then we would not have a channel. Hence $\theta\neq 0$ and so

$$9-4x+2x\cos(\theta)=0$$

$$\cos(\theta)=\frac{4x-9}{2x} \qquad (*)$$

Differentiating partially with respect to θ:

$$A=(9-2x)x\sin(\theta)+x^2\sin(\theta)\cos(\theta)$$

$$\frac{\partial A}{\partial \theta}=(9-2x)x\cos(\theta)+x^2\left[\cos^2(\theta)-\sin^2(\theta)\right]$$

$$=(9-2x)x\cos(\theta)+x^2\left[\cos^2(\theta)-\left(1-\cos^2(\theta)\right)\right]$$

$$=(9-2x)x\cos(\theta)+x^2\left[2\cos^2(\theta)-1\right]$$

Substituting $\cos(\theta)=\frac{4x-9}{2x}$ and putting $\frac{\partial A}{\partial \theta}=0$ gives

$$(9-2x)x\left(\frac{4x-9}{2x}\right)+x^2\left[2\left(\frac{4x-9}{2x}\right)^2-1\right]=\frac{1}{2}(9-2x)(4x-9)+\frac{1}{2}(4x-9)^2-x^2$$

$$=\frac{4x-9}{2}\left[9-2x+4x-9\right]-x^2$$

$$=x(4x-9)-x^2$$

$$=4x^2-9x-x^2=3x^2-9x=0$$

Factorizing we have

$$3x(x-3)=0$$

$$x=0 \text{ or } x=3$$

Again $x=0$ gives no channel so $x=3m$. To find θ we put $x=3$ into $\cos(\theta)=\frac{4x-9}{2x}$

$$\cos(\theta)=\frac{4(3)-9}{2(3)}=\frac{1}{2} \text{ gives } \theta=60^\circ$$

To check maximum channel capacity we have to find the second partial derivatives.

$$\frac{\partial A}{\partial x} = \left[9 - 4x + 2x\cos(\theta)\right]\sin(\theta)$$

$$\frac{\partial^2 A}{\partial x^2} = \left[-4 + 2\cos(\theta)\right]\sin(\theta)$$

$$\frac{\partial A}{\partial \theta} = (9 - 2x)x\cos(\theta) + x^2\left[2\cos^2(\theta) - 1\right]$$

$$\frac{\partial^2 A}{\partial \theta^2} = (9 - 2x)x\left[-\sin(\theta)\right] + x^2\left[4\cos(\theta)\left[-\sin(\theta)\right]\right]$$

Also

$$\frac{\partial^2 A}{\partial x \partial \theta} = \frac{\partial}{\partial x}\left[\left(9x - 2x^2\right)\cos(\theta) + 2x^2\cos^2(\theta) - x^2\right]$$

$$= (9 - 4x)\cos(\theta) + 4x\cos^2(\theta) - 2x$$

Substituting $\theta = 60°$ and $x = 3$ gives

$$\left(\frac{\partial^2 A}{\partial x^2}\right)\left(\frac{\partial^2 A}{\partial \theta^2}\right) - \left(\frac{\partial^2 A}{\partial x \partial \theta}\right)^2 > 0$$

Substituting $\theta = 60°$ into $\frac{\partial^2 A}{\partial x^2} = \left[-4 + 2\cos(\theta)\right]\sin(\theta)$ gives

$$\frac{\partial^2 A}{\partial x^2} = \left[-4 + 2\cos(60°)\right]\sin(60°) = -3\frac{\sqrt{3}}{2} < 0$$

Hence $x = 3$ m and $\theta = 60°$ gives maximum channel capacity.