

Complete Solutions to Examination Questions 10

1. (i) We need to determine $3z_1 + 2z_2$ given $z_1 = 4 + j2$, $z_2 = 3 - j$:

$$\begin{aligned} 3z_1 + 2z_2 &= 3(4 + j2) + 2(3 - j) \\ &= 12 + j6 + 6 - 2j = 18 + j4 \end{aligned}$$

(ii) The product $z_1 z_2$ is given by

$$\begin{aligned} z_1 z_2 &= (4 + j2)(3 - j) \\ &= 12 - j4 + j6 - j^2 2 && \text{[Using FOIL]} \\ &= 12 + j2 + 2 && \text{[Because } j^2 = -1\text{]} \\ &= 14 + j2 \end{aligned}$$

2. (a) We need to find $z_1 z_2$ given $z_1 = 2 + j$, $z_2 = 3 - j4$:

$$\begin{aligned} z_1 z_2 &= (2 + j)(3 - j4) \\ &= 6 - j8 + j3 - j^2 4 && \text{[Using FOIL]} \\ &= 6 - j5 + 4 && \text{[Because } j^2 = -1\text{]} \\ &= 10 - j5 \end{aligned}$$

(b) How do we find the square root of $z_1 = 2 + j$?

We first write $2 + j$ in polar form:

$$\begin{aligned} 2 + j &= \sqrt{2^2 + 1^2} \angle \left(\tan^{-1} \left(\frac{1}{2} \right) \right) \\ &= \sqrt{5} \angle (26.56^\circ) = 5^{1/2} \angle (26.56^\circ) \end{aligned}$$

The square root of $5^{1/2} \angle (26.56^\circ)$ is given by De-Moivre's Theorem (if $z = r \angle \theta$ then $z^n = r^n \angle (n\theta)$):

$$\begin{aligned} [5^{1/2} \angle (26.56^\circ)]^{1/2} &= (5^{1/2})^{1/2} \angle \left(\frac{1}{2} \times 26.56^\circ \right) \\ &= 5^{1/4} \angle (13.28^\circ) \end{aligned}$$

The other root has the same modulus but we add 180° because $\frac{360}{2} = 180^\circ$

$$5^{1/4} \angle (13.28^\circ + 180^\circ) = 5^{1/4} \angle (193.28^\circ)$$

The two roots of z_1 are $5^{1/4} \angle (13.28^\circ)$ and $5^{1/4} \angle (193.28^\circ)$.

3. (a) We need to write $z^4 = -2 - j2\sqrt{3}$ in polar form and then take the fourth root of this:

$$\begin{aligned} -2 - j2\sqrt{3} &= \sqrt{(-2)^2 + (-2\sqrt{3})^2} \angle \left(\tan^{-1} \left(\frac{2\sqrt{3}}{2} \right) \right) \\ &= 4 \angle (180^\circ + 60^\circ) = 4 \angle (240^\circ) \end{aligned}$$

Remember $-2 - j2\sqrt{3}$ is in the third quadrant and that is why we had to add 180° .

What do we need to find?

The four roots of this number. Let z_1 be the first root

$$\begin{aligned} z_1 &= [-2 - j2\sqrt{3}]^{1/4} \\ &= [4\angle(240^\circ)]^{1/4} \\ &= 4^{1/4} \angle\left(\frac{1}{4} \times 240^\circ\right) = \sqrt{2} \angle(60^\circ) \quad \left[\text{Because } 4^{1/4} = (2^2)^{1/4} = 2^{1/2} = \sqrt{2} \right] \end{aligned}$$

One of the roots is $z_1 = \sqrt{2} \angle(60^\circ)$. How many more roots do we have?

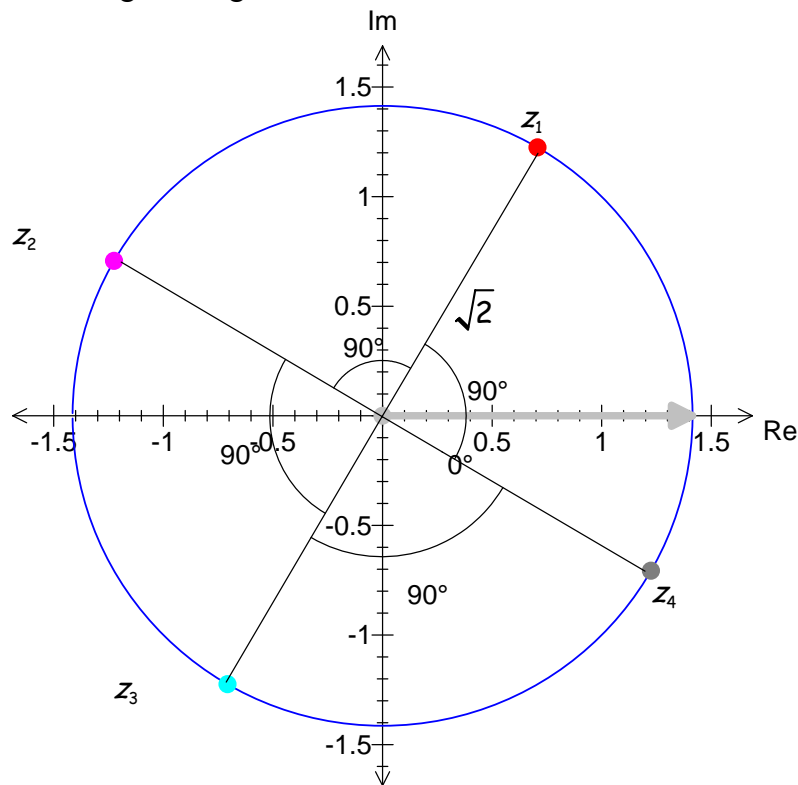
Three more. How do we find these?

By adding $\frac{360}{4} = 90^\circ$ each time to the argument of $z_1 = \sqrt{2} \angle(60^\circ)$.

$$z_2 = \sqrt{2} \angle(60^\circ + 90^\circ), \quad z_3 = \sqrt{2} \angle(60^\circ + 2(90^\circ)), \quad z_4 = \sqrt{2} \angle(60^\circ + 3(90^\circ))$$

$$z_2 = \sqrt{2} \angle(150^\circ), \quad z_3 = \sqrt{2} \angle(240^\circ) \quad \text{and} \quad z_4 = \sqrt{2} \angle(330^\circ)$$

Plotting these on the Argand diagram:



(b) We need to place each of these numbers into polar form:

$$\cos(53^\circ) + j \sin(53^\circ) = 1 \angle(53^\circ)$$

$$\cos(24^\circ) + j \sin(24^\circ) = 1 \angle(24^\circ)$$

Evaluating the given expression

$$\begin{aligned} \frac{(\cos 53^\circ + j \sin 53^\circ)^3}{\cos 24^\circ + j \sin 24^\circ} &= \frac{[1 \angle (53^\circ)]^3}{1 \angle (24^\circ)} \\ &= \frac{(1)^3 \angle (3 \times 53^\circ)}{1 \angle (24^\circ)} \quad \left[\text{Using } [r \angle \theta]^n = r^n \angle (n\theta) \right] \\ &= (1)^2 \angle [(3 \times 53^\circ) - 24^\circ] \quad \left[\text{Using } \frac{r \angle \theta}{q \angle \beta} = \frac{r}{q} \angle (\theta - \beta) \right] \\ &= 1 \angle (135^\circ) \end{aligned}$$

Our final answer is $1 \angle 135^\circ$.

4. (a) We need to solve the quadratic equation $x^2 - 6x + 13 = 0$. *How?*
Use the quadratic formula with $a = 1$, $b = -6$ and $c = 13$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - (4 \times 1 \times 13)}}{2 \times 1} \\ &= \frac{6 \pm \sqrt{-16}}{2} = \frac{6}{2} \pm j \frac{4}{2} = 3 \pm j2 \end{aligned}$$

Our roots with imaginary number $i = j$ we have $3 + 2i$, $3 - 2i$.

(b) The exponential form of a complex number is

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

With $\theta = \frac{\pi}{4}$ we have

$$e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1 + i)$$

(c) De Moivre's theorem says that if $z = e^{i\theta}$ then

$$\begin{aligned} z^n &= e^{in\theta} \\ &= \cos(n\theta) + i \sin(n\theta) \end{aligned}$$

We have

$$\begin{aligned} z^n + \frac{1}{z^n} &= e^{in\theta} + \frac{1}{e^{in\theta}} \\ &= e^{in\theta} + e^{-in\theta} \quad \left[\text{Using the rules of indices } \frac{1}{a^n} = a^{-n} \right] \\ &= \cos(n\theta) + i \sin(n\theta) + \underbrace{\cos(n\theta) - i \sin(n\theta)}_{= e^{-in\theta}} \\ &= 2 \cos(n\theta) \end{aligned}$$

The value of a is 2.

5. We use the complex numbers $a = 3 + i$, $b = 2 - 3i$, $c = -1 + 4i$:

(a) (i) Evaluating

$$\begin{aligned}(a+b)c &= (3+i+2-3i)(-1+4i) \\ &= (5-2i)(-1+4i) \\ &= -5+20i+2i-8i^2 \quad [\text{Using FOIL}] \\ &= -5+22i+8 = 3+22i\end{aligned}$$

(ii) The $(bc)^*$ means determine the complex conjugate of bc . We first find bc :

$$\begin{aligned}bc &= (2-3i)(-1+4i) \\ &= -2+8i+3i-12i^2 = 10+11i\end{aligned}$$

The complex conjugate of $bc = 10+11i$ is

$$(bc)^* = 10-11i$$

(iii) Remember to divide two complex numbers we have to multiply the numerator and denominator by the complex conjugate of the denominator.

$$\begin{aligned}\frac{b}{a} &= \frac{2-3i}{3+i} = \frac{(2-3i)(3-i)}{3^2+1^2} \\ &= \frac{6-2i-9i+3i^2}{10} = \frac{3-11i}{10} = 0.3-1.1i\end{aligned}$$

(iv) What does the notation $|ab|$ mean?

The modulus of ab . We first find the product ab and then the modulus.

$$\begin{aligned}ab &= (3+i)(2-3i) = 6-9i+2i-3i^2 \\ &= 9-7i \\ |9-7i| &= \sqrt{9^2+7^2} = \sqrt{130}\end{aligned}$$

(v) What does the notation $\text{Re}(ac)$ mean?

Find the real part of the product ac . We have

$$\begin{aligned}ac &= (3+i)(-1+4i) \\ &= -3+12i-i+4i^2 = -7+11i\end{aligned}$$

The real part is equal to -7 .

(vi) Need to convert each of the given complex numbers into polar form:

$$\begin{aligned}|a| &= |3+i| = \sqrt{3^2+1^2} = \sqrt{10} \\ \arg(a) &= \tan^{-1}\left(\frac{1}{3}\right) = 0.3218\end{aligned}$$

Therefore a in polar form is $\sqrt{10}\angle(0.3218)$. Similarly we have

$$|b| = |2-3i| = \sqrt{2^2+(-3)^2} = \sqrt{13}, \quad \arg(b) = \tan^{-1}\left(-\frac{3}{2}\right) = -0.9828$$

$$|c| = |-1+4i| = \sqrt{(-1)^2+4^2} = \sqrt{17}, \quad \arg(c) = \tan^{-1}(-4) = -1.3258 + \pi = 1.8158$$

We need to add π radians to the argument of c because the calculator gives the wrong quadrant. We have $b = \sqrt{13}\angle(-0.9828)$ and $c = \sqrt{17}\angle(1.8158)$.

Therefore

$$\begin{aligned} \left(\frac{ac}{b}\right) &= \frac{\sqrt{10}\angle(0.3218) \times \sqrt{17}\angle(1.8158)}{\sqrt{13}\angle(-0.9828)} \\ &= \frac{\sqrt{10}\sqrt{17}}{\sqrt{13}} \angle(0.3218 + 1.8158 - (-0.9828)) \\ &= 3.6162 \angle(3.1204) \end{aligned}$$

The modulus is 3.6162 and argument 3.1204.

(b) To apply De Moivre's theorem we first have to convert the given complex number into polar form. We can find the argument in degrees so that we get a feel for the size of the angle.

$$\begin{aligned} 3-i &= \sqrt{3^2 + (-1)^2} \angle\left(\tan^{-1}\left(-\frac{1}{3}\right)\right) \\ &= \sqrt{10} \angle(-18.43^\circ) \end{aligned}$$

What does De Moivre's theorem state?

If $z = r\angle\theta$ then $z^n = r^n\angle(n\theta)$. We have

$$\begin{aligned} (3-i)^{1/3} &= \left[\sqrt{10}\angle(-18.43)\right]^{1/3} \\ &= (\sqrt{10})^{1/3} \angle\left(-18.43 \times \frac{1}{3}\right) \\ &= (10^{1/2})^{1/3} \angle(-6.145^\circ) = 10^{1/6} \angle(-6.145^\circ) \quad \left[\text{Because } (a^m)^n = a^{mn}\right] \end{aligned}$$

One of the roots of $(3-i)^{1/3}$ is $10^{1/6} \angle(-6.145^\circ) = r_1$. How many roots are there?

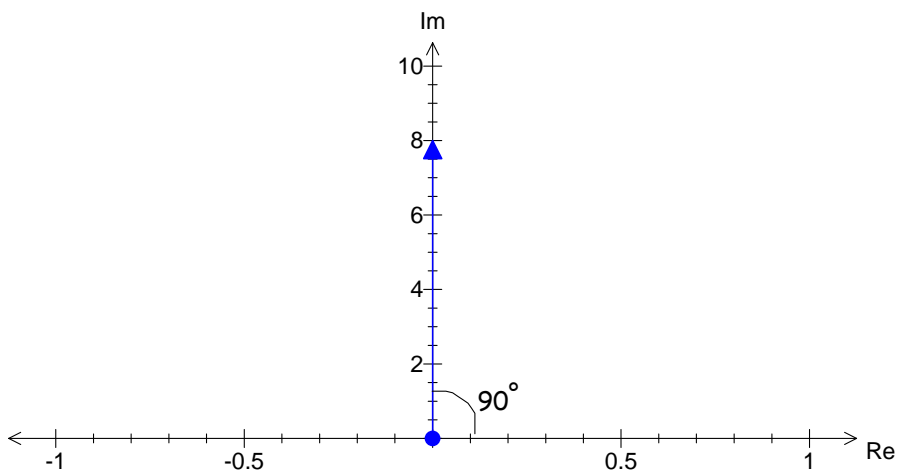
Three roots because we are asked to find the cube roots. How do we find the other two roots?

Add $\left(\frac{360}{3}\right)^\circ = 120^\circ$ each time. Let r_2 and r_3 be the other two roots:

$$r_2 = 10^{1/6} \angle(-6.145^\circ + 120^\circ) = 10^{1/6} \angle(113.855^\circ)$$

$$r_3 = 10^{1/6} \angle(113.855^\circ + 120^\circ) = 10^{1/6} \angle(233.855^\circ)$$

6. We first convert $8i$ into polar form. On the Argand diagram $8i$ is plotted as:



Clearly by examining this diagram we have $8i = 8\angle(90^\circ)$. The cube roots of this number are given by taking the number to the index $1/3$. Let z_1 be the first root then

$$\begin{aligned} z_1 &= [8\angle(90^\circ)]^{1/3} = 8^{1/3} \angle\left(\frac{1}{3} \times 90^\circ\right) \\ &= 2\angle(30^\circ) \end{aligned}$$

We have two other roots z_2 and z_3 . *Where are these?*

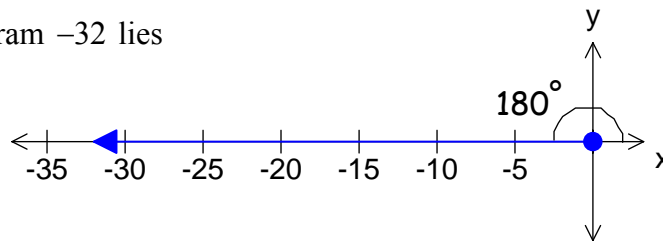
These have a modulus 2 but the argument of $\frac{360}{3} = 120^\circ$ is added each time to z_1 and then to z_2 :

$$z_2 = 2\angle(30^\circ + 120^\circ) = 2\angle(150^\circ) \text{ and } z_3 = 2\angle(150^\circ + 120^\circ) = 2\angle(270^\circ)$$

The three roots of $8i$ are $z_1 = 2\angle(30^\circ)$, $z_2 = 2\angle(150^\circ)$ and $z_3 = 2\angle(270^\circ)$.

7. We need to find the 5th roots of -32 because we are given $z^5 = -32$. *What is -32 in polar form?*

On the Argand diagram -32 lies



-32 has a modulus of 32 and argument of 180° , that is

$$-32 = 32\angle(180^\circ)$$

We can find one of the roots of -32 by taking this $32\angle(180^\circ)$ to the index $1/5$. Let z_1 be one of the roots then

$$\begin{aligned} z_1 &= [32\angle(180^\circ)]^{1/5} \\ &= 32^{1/5} \angle\left(\frac{1}{5} \times 180^\circ\right) = 2\angle(36^\circ) \quad \left[\text{Because } z^n = r^n \angle(n\theta)\right] \end{aligned}$$

The other four roots are found by adding $\frac{360}{5} = 72^\circ$ to each root which results in

$$z_2 = 2\angle(36^\circ + 72^\circ) = 2\angle(108^\circ)$$

$$z_3 = 2\angle(108^\circ + 72^\circ) = 2\angle(180^\circ)$$

$$z_4 = 2\angle(180^\circ + 72^\circ) = 2\angle(252^\circ)$$

$$z_5 = 2\angle(252^\circ + 72^\circ) = 2\angle(324^\circ)$$

8. (a) We need to convert $w = 3e^{3\pi i/4}$ into rectangular form:

$$\begin{aligned} w &= 3e^{3\pi i/4} = 3 \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \\ &= 3 \left[-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right] = \frac{3}{\sqrt{2}} [-1 + i] \quad \left[\text{Taking out } \frac{1}{\sqrt{2}} \right] \end{aligned}$$

The product zw is given by

$$\begin{aligned}
zw &= (3+5i)\frac{3}{\sqrt{2}}(-1+i) \\
&= \frac{3}{\sqrt{2}}(-3+3i-5i+5i^2) \\
&= \frac{3}{\sqrt{2}}(-8-2i) = -\frac{3(2)}{\sqrt{2}}(4+i) && \left[\text{Because } -8-2i = -2(4+i) \right] \\
&= -3\sqrt{2}(4+i) && \left[\text{Because } \frac{2}{\sqrt{2}} = \frac{2}{2^{1/2}} = 2^{1/2} = \sqrt{2} \right]
\end{aligned}$$

The quotient $\frac{z}{w}$ is given by

$$\begin{aligned}
\frac{z}{w} &= \frac{3+5i}{\frac{3}{\sqrt{2}}(-1+i)} = \frac{\sqrt{2}(3+5i)(-1-i)}{3(-1)^2+1^2} && \left[\text{Complex Conjugate of} \right. \\
& && \left. -1+i \text{ is } -1-i \right] \\
&= \frac{\sqrt{2}}{3} \left[\frac{-3-3i-5i-5i^2}{2} \right] \\
&= \frac{\sqrt{2}}{3} \left[\frac{2-8i}{2} \right] = \frac{\sqrt{2}}{3} [1-4i] && \left[\text{Because } \frac{2-8i}{2} = \frac{2(1-4i)}{2} \right]
\end{aligned}$$

The addition $z+w$ is equal to

$$\begin{aligned}
z+w &= (3+5i) + \frac{3}{\sqrt{2}}(-1+i) \\
&= \left(3 - \frac{3}{\sqrt{2}} \right) + \left(5 + \frac{3}{\sqrt{2}} \right) i = 0.8787 + 7.1213i
\end{aligned}$$

We now place each of these results in polar form:

$$\begin{aligned}
zw &= -3\sqrt{2}(4+i) = 3\sqrt{2}(-4-i) = 3\sqrt{2} \left[\sqrt{4^2+1^2} \angle \left(\tan^{-1} \left(\frac{-1}{-4} \right) \right) \right] \\
&= 3\sqrt{2}\sqrt{17} \angle (0.245 + \pi) = 3\sqrt{34} \angle (3.3866)
\end{aligned}$$

Since zw is in the third quadrant that is why we have added π radians.

The quotient result

$$\begin{aligned}
\frac{z}{w} &= \frac{\sqrt{2}}{3} [1-4i] = \frac{\sqrt{2}}{3} \sqrt{17} \angle (\tan^{-1}(-4)) \\
&= \frac{\sqrt{34}}{3} \angle (-1.3258)
\end{aligned}$$

The addition $z+w$ in polar form is

$$z+w = 0.8787 + 7.1213i = 7.1753 \angle (1.448)$$

(b) Let $z = a+bi$ where a and b are real. Substituting this into the given equation

$z^2 + 2\bar{z} = -3$ yields:

$$(a+bi)^2 + 2(a-bi) = -3 \quad \left[\text{Remember if } z = a+ib \text{ then } \bar{z} = a-ib \right]$$

$$a^2 + 2abi + (bi)^2 + 2a - 2bi = -3 \quad \left[\text{Expanding} \right]$$

$$a^2 - b^2 + 2a + (2ab - 2b)i = -3 \quad \left[\text{Remember } (bi)^2 = i^2b^2 = -b^2 \right]$$

Equating real and imaginary parts gives

$$a^2 - b^2 + 2a = -3 \quad \text{and} \quad 2ab - 2b = 0 \Rightarrow 2b(a-1) = 0$$

From the last equation $2b(a-1) = 0$ we have $b = 0$ or $a = 1$. If $b = 0$ then the first equation $a^2 - b^2 + 2a = -3$ becomes $a^2 + 2a = -3$ which implies $a^2 + 2a + 3 = 0$. This quadratic equation gives us complex roots but we are assuming that a is real. Hence b **cannot equal** zero, that is $b \neq 0$ which means that $a = 1$. Substituting $a = 1$ into $a^2 - b^2 + 2a = -3$ gives

$$1 - b^2 + 2 = -3$$

$$6 = b^2 \Rightarrow b = \pm\sqrt{6}$$

Our z values are $z = a + bi = 1 + \sqrt{6}i, 1 - \sqrt{6}i$.

(c) In polar form we have

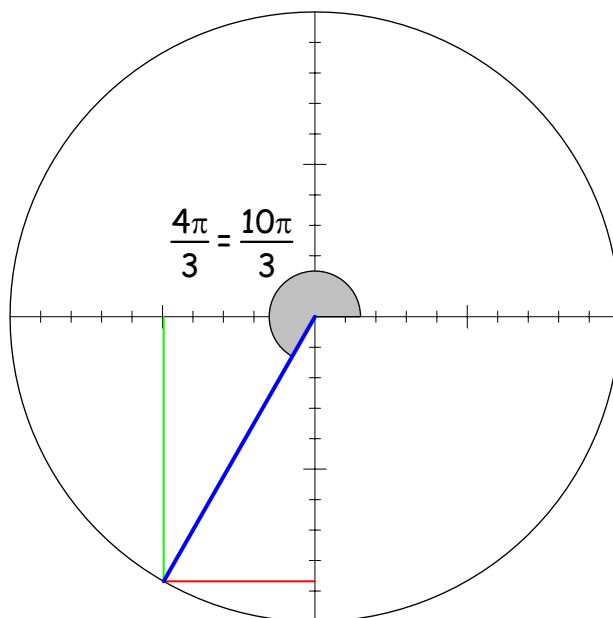
$$\begin{aligned} z = 1 + \sqrt{3}i &= \sqrt{1^2 + (\sqrt{3})^2} \angle \left(\tan^{-1} \left(\frac{\sqrt{3}}{1} \right) \right) \\ &= 2 \angle \left(\frac{\pi}{3} \right) \end{aligned}$$

How do we find z^{10} ?

Use De Moivre's Theorem which states that if $z = r \angle \theta$ then $z^n = r^n \angle (n\theta)$:

$$z^{10} = \left(2 \angle \left(\frac{\pi}{3} \right) \right)^{10} = 2^{10} \angle \left(\frac{10\pi}{3} \right) = 1024 \angle \left(\frac{4\pi}{3} \right)$$

Remember $\frac{10\pi}{3} = 2\pi + \frac{4\pi}{3}$ radians is $\frac{4\pi}{3}$ radians because 2π radians is a complete revolution which means we are back at our starting point.



What does $\sqrt[3]{z} = z^{1/3}$ mean?

The cube roots of z . *How many are there?*

Three. We can find the first root by applying De Moivre's Theorem:

$$z^{1/3} = \left[2 \angle \left(\frac{\pi}{3} \right) \right]^{1/3} = 2^{1/3} \angle \left(\frac{1}{3} \times \frac{\pi}{3} \right) = 2^{1/3} \angle \left(\frac{\pi}{9} \right) = z_1$$

We have two more roots and the other roots have exactly the same modulus but argument is increased by $\frac{2\pi}{3}$ each time:

$$z_2 = 2^{1/3} \angle \left(\frac{\pi}{9} + \frac{2\pi}{3} \right) = 2^{1/3} \angle \left(\frac{7\pi}{9} \right)$$

$$z_3 = 2^{1/3} \angle \left(\frac{7\pi}{9} + \frac{2\pi}{3} \right) = 2^{1/3} \angle \left(\frac{13\pi}{9} \right)$$

(d) Using the hint means we need to expand $(\cos(\theta) + i \sin(\theta))^3$. Apply Binomial expansion:

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^3 &= \cos^3(\theta) + 3\cos^2(\theta)i \sin(\theta) + 3\cos(\theta)i^2 \sin^2(\theta) + i^3 \sin^3(\theta) \\ &= \cos^3(\theta) + i3\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta) \\ &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i[3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)] \quad (\dagger) \end{aligned}$$

Collecting real and imaginary terms in the last line.

We can write $\cos(\theta) + i \sin(\theta) = 1 \angle \theta$. Raising this to the index 3 gives

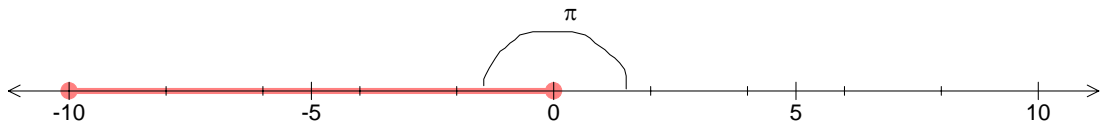
$$\begin{aligned} [\cos(\theta) + i \sin(\theta)]^3 &= (1 \angle \theta)^3 \\ &= 1^3 \angle (3\theta) \\ &= 1[\cos(3\theta) + i \sin(3\theta)] = \cos(3\theta) + i \sin(3\theta) \quad (*) \end{aligned}$$

Equating the real and imaginary parts of (\dagger) and $(*)$ gives

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$

9. Need to be careful with this question because of minus sign in front of the 10. *How can we write the minus 10 in polar form?*

Remember -10 on the Argand diagram lies



Therefore -10 in polar form is $10e^{i\pi}$ because modulus is 10 and argument is π . We have

$$\begin{aligned} z &= -10e^{-i\pi/10} = 10e^{i\pi} e^{-i\pi/10} \\ &= 10e^{i\left(\pi - \frac{\pi}{10}\right)} \quad \left[\text{Applying the rules of indices } a^m a^n = a^{m+n} \right] \\ &= 10e^{i\left(\frac{9\pi}{10}\right)} \end{aligned}$$

The argument of the complex number is $\frac{9\pi}{10}$.

10. (a) We are given the complex numbers $z = 2 - j$ and $w = 3 + 2j$. We need to determine

$$\bar{z}, \quad z + w, \quad zw, \quad \frac{1}{w}, \quad |z|$$

What does \bar{z} denote?

The conjugate of z which means that $\bar{z} = 2 + j$.

Adding the two complex numbers gives

$$z + w = (2 - j) + (3 + 2j) = 5 + j$$

The product zw can be found by using FOIL:

$$\begin{aligned} zw &= (2 - j)(3 + 2j) = 6 + 4j - 3j - j^2 \\ &= 6 + j + 1 = 7 + j \end{aligned}$$

The reciprocal $1/w$ is given by

$$\frac{1}{w} = \frac{1}{3 + 2j} = \frac{3 - 2j}{3^2 + 2^2} = \frac{3 - 2j}{13} = \frac{3}{13} - \frac{2}{13}j$$

What does $|z|$ denote?

This is the modulus of z . We have

$$|z| = |2 - j| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

(b) Need to write $z = -2 - 2\sqrt{3}j$ in modulus argument form:

$$|z| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\arg(z) = \arg(-2 - 2\sqrt{3}j) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

Remember $z = -2 - 2\sqrt{3}j$ is in the third quadrant therefore we have added π radians.

z in modulus argument form is $4\angle\left(\frac{4\pi}{3}\right)$.

We are told that $w^2 = -2 - 2\sqrt{3}j$ therefore $w = (-2 - 2\sqrt{3}j)^{1/2}$. We apply De Moivre's theorem to find the square root of w . De Moivre's theorem states that if $z = r\angle\theta$ then $z^n = r^n\angle(n\theta)$. Let w_1 be one of the roots then using the polar form of z we have

$$w_1 = \left[4\angle\left(\frac{4\pi}{3}\right)\right]^{1/2} = 4^{1/2}\angle\left(\frac{1}{2} \times \frac{4\pi}{3}\right) = 2\angle\left(\frac{2\pi}{3}\right)$$

We have two square roots of w . *Where is the other root?*

Same modulus but the argument of π radians is added because we have a square root which means we add $\frac{2\pi}{2} = \pi$ radians.

$$w_2 = 2\angle\left(\frac{2\pi}{3} + \pi\right) = 2\angle\left(\frac{5\pi}{3}\right)$$

The two values of w are $w_1 = 2\angle\left(\frac{2\pi}{3}\right)$ and $w_2 = 2\angle\left(\frac{5\pi}{3}\right)$.

11. (a) Using FOIL we have

$$\begin{aligned}(2+3j)(1-j) &= 2 - j2 + j3 - j^23 \\ &= 2 + j + 3 = 5 + j\end{aligned}$$

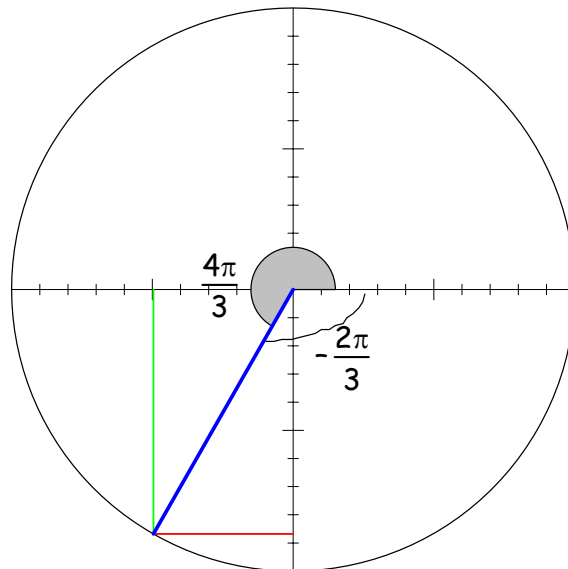
For the quotient we have to multiply the numerator and denominator by the complex conjugate of $1-j$ which is $1+j$.

$$\begin{aligned}\frac{2+3j}{1-j} &= \frac{(2+3j)(1+j)}{1^2+1^2} = \frac{2+2j+3j+j^23}{2} \\ &= \frac{2+5j-3}{2} = -\frac{1}{2} + \frac{5}{2}j\end{aligned}$$

(b) This is the same complex number as part (b) of question 10. From that solution we have

$$z = -2 - 2\sqrt{3}j = 4 \angle \left(\frac{4\pi}{3} \right) = 4 \angle \left(-\frac{2\pi}{3} \right)$$

The $\frac{4\pi}{3}$ is **not** the principal argument but $-\frac{2\pi}{3}$ is the principal argument because the principal argument lies between $-\pi$ and π .



(c) Remember -25 has a modulus of 25 and an argument of π radians.

$$-25 = 25 \angle (\pi)$$

The given quartic equation $w^4 + 25 = 0$ can be rearranged to

$$w^4 = -25$$

$$w = (-25)^{1/4}$$

We have four roots of -25 . Let w_1 be the first root. By applying De Moivre's theorem we have

$$\begin{aligned}
 w_1 &= (-25)^{1/4} = (25 \angle (\pi))^{1/4} \\
 &= 25^{1/4} \angle \left(\frac{1}{4} \times \pi \right) \quad \left[\text{Because } (r \angle \theta)^n = r^n \angle (n\theta) \right] \\
 &= (5^2)^{1/4} \angle \left(\frac{\pi}{4} \right) = \sqrt{5} \angle \left(\frac{\pi}{4} \right) \quad \left[\text{Because } (5^2)^{1/4} = 5^{1/2} = \sqrt{5} \right]
 \end{aligned}$$

How do we find the other three roots?

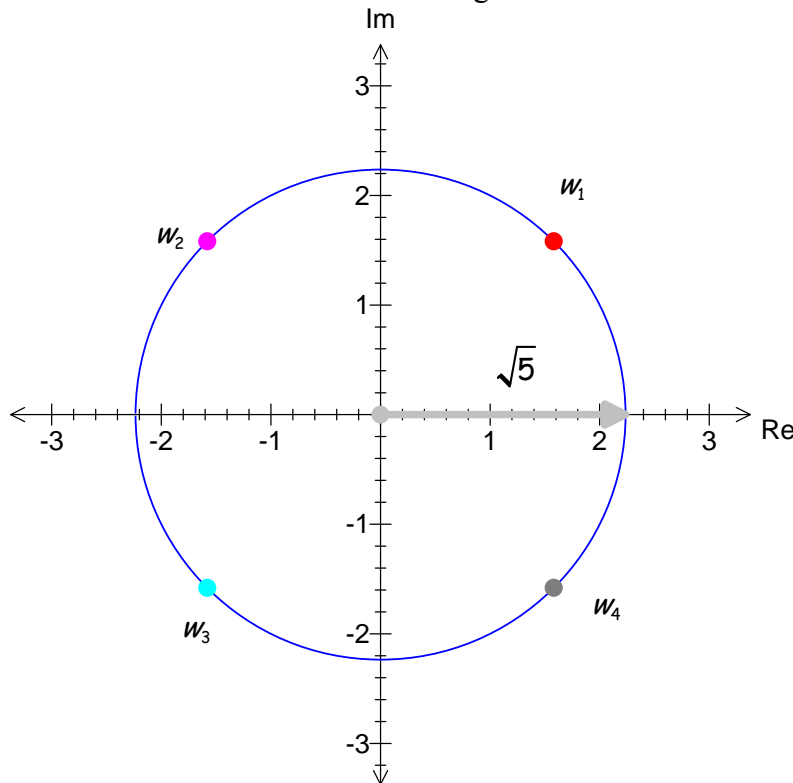
Add $\frac{2\pi}{4} = \frac{\pi}{2}$ radians each time. Let w_2 , w_3 and w_4 be the other three roots.

$$w_2 = \sqrt{5} \angle \left(\frac{\pi}{4} + \frac{\pi}{2} \right) = \sqrt{5} \angle \left(\frac{3\pi}{4} \right)$$

$$w_3 = \sqrt{5} \angle \left(\frac{3\pi}{4} + \frac{\pi}{2} \right) = \sqrt{5} \angle \left(\frac{5\pi}{4} \right)$$

$$w_4 = \sqrt{5} \angle \left(\frac{5\pi}{4} + \frac{\pi}{2} \right) = \sqrt{5} \angle \left(\frac{7\pi}{4} \right)$$

All these roots lie on the circle with centre origin and radius $\sqrt{5}$ as illustrated below:



12. (a) The complex number $1-i$ in polar form is

$$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \arg(1-i) = \tan^{-1}(-1) = -45^\circ \text{ or } -\frac{\pi}{4}$$

Therefore $1-i = \sqrt{2} \angle \left(-\frac{\pi}{4} \right) = \sqrt{2} e^{-i\frac{\pi}{4}}$.

(b) The complex number $e^{i\pi}$ in rectangular form is

$$e^{i\pi} = \cos(\pi) - i \sin(\pi) = -1 - 0 = -1$$

(c) We first write the complex number $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ in polar form:

$$\left| \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$$

$$\arg\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \tan^{-1}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right) = \tan^{-1}(1) = 45^\circ$$

We have $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = 1\angle(45^\circ)$. Therefore

$$\begin{aligned} \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^{60} &= [1\angle(45^\circ)]^{60} \\ &= (1)^{60} \angle(60 \times 45^\circ) \\ &= 1\angle(2700^\circ) = 1\angle(180^\circ) = -1 \end{aligned}$$

(d) We can write 8 in polar form as $8\angle(0^\circ)$. The cube roots of 8 are given by

$$8^{1/3} = [8\angle(0^\circ)]^{1/3} = 8^{1/3} \angle\left(\frac{1}{3} \times 0^\circ\right) = 2\angle(0^\circ) = r_1$$

The other roots are found by adding $\frac{360}{3} = 120^\circ$:

$$r_2 = 2\angle(0^\circ + 120^\circ) = 2\angle(120^\circ)$$

$$r_3 = 2\angle(120^\circ + 120^\circ) = 2\angle(240^\circ)$$

The cube roots of 8 are $r_1 = 2\angle(0^\circ)$, $r_2 = 2\angle(120^\circ)$ and $r_3 = 2\angle(240^\circ)$.