Complete Solutions to Examination Questions 12

1. We are given $\mathbf{r} = 2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $\mathbf{s} = 4\mathbf{i}-5\mathbf{j}-2\mathbf{k}$ and we need to determine $\mathbf{r}\times\mathbf{s}$. How?
Use (12.24):

$$\begin{aligned}\mathbf{r}\times\mathbf{s} = (2\mathbf{i}+3\mathbf{j}-4\mathbf{k})\times(4\mathbf{i}-5\mathbf{j}-2\mathbf{k}) &= \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & -4\\ 4 & -5 & -2\end{pmatrix}\\ &= \mathbf{i}\det\begin{pmatrix}3 & -4\\ -5 & -2\end{pmatrix} - \mathbf{j}\det\begin{pmatrix}2 & -4\\ 4 & -2\end{pmatrix} + \mathbf{k}\det\begin{pmatrix}2 & 3\\ 4 & -5\end{pmatrix}\\ &= \mathbf{i}(-6-20) - \mathbf{j}(-4+16) + \mathbf{k}(-10-12)\\ &= -26\mathbf{i} - 12\mathbf{j} - 22\mathbf{k}\end{aligned}$$

2. (a) (i) To find the work done, we need to find the scalar product $\mathbf{F}\cdot\mathbf{r}$:

$$\begin{aligned}\mathbf{F}\cdot\mathbf{r} &= (6\mathbf{i}+2\mathbf{j}-5\mathbf{k})\cdot(9\mathbf{i}+6\mathbf{j}+\mathbf{k})\\ &= (6\times 9)+(2\times 6)+(-5\times 1) = 61\text{J (J = Joule)}\end{aligned}$$

(ii) Let the angle between $\mathbf{F}$ and $\mathbf{r}$ be θ. How do we find θ?

Use (12.14) $\cos(\theta) = \dfrac{\mathbf{F}\cdot\mathbf{r}}{|\mathbf{F}||\mathbf{r}|}$. What is $|\mathbf{F}|$ and $|\mathbf{r}|$ equal to?

$$|\mathbf{F}| = |6\mathbf{i}+2\mathbf{j}-5\mathbf{k}| = \sqrt{6^2+2^2+(-5)^2} = \sqrt{65}$$

$$|\mathbf{r}| = |9\mathbf{i}+6\mathbf{j}+\mathbf{k}| = \sqrt{9^2+6^2+1^2} = \sqrt{118}$$

Substituting these $\mathbf{F}\cdot\mathbf{r} = 61$ and $|\mathbf{F}| = \sqrt{65},\ |\mathbf{r}| = \sqrt{118}$ into $\cos(\theta) = \dfrac{\mathbf{F}\cdot\mathbf{r}}{|\mathbf{F}||\mathbf{r}|}$ gives

$$\cos(\theta) = \frac{\mathbf{F}\cdot\mathbf{r}}{|\mathbf{F}||\mathbf{r}|} = \frac{61}{\sqrt{65}\sqrt{118}}$$

Taking inverse cosine results in $\theta = \cos^{-1}\left(\dfrac{61}{\sqrt{65}\sqrt{118}}\right) = 45.85°$.

(b) (i) We need to find the velocity and acceleration in terms of time by differentiating:

$$\text{velocity} = \dot{\mathbf{r}} = (3t^2-6t)\mathbf{i} + (12t-2)\mathbf{j}$$

$$\text{acceleration} = \ddot{\mathbf{r}} = (6t-6)\mathbf{i} + 12\mathbf{j}$$

(ii) Initial velocity and acceleration is found by substituting $t = 0$ into the results of part (b) (i):

$$\text{initial velocity} = \left[3(0)^2-(6\times 0)\right]\mathbf{i} + \left[12(0)-2\right]\mathbf{j} = -2\mathbf{j}$$

$$\text{initial acceleration} = \left[6(0)-6\right]\mathbf{i} + 12\mathbf{j} = -6\mathbf{i}+12\mathbf{j}$$

(iii) The horizontal velocity is zero when the coefficient of $\mathbf{i}$ in $(3t^2-6t)\mathbf{i}+(12t-2)\mathbf{j}$ is zero:

$$\begin{aligned}&3t^2-6t = 0\\ &3t(t-2) = 0 \quad\Rightarrow\quad t = 0,\quad t = 2\end{aligned}$$

The horizontal velocity is zero at $t = 0,\quad t = 2$.

3. We have to determine $\mathbf{a}\times\mathbf{b}$ given $\mathbf{a} = (3, 5, -2)$ and $\mathbf{b} = (2, 4, 7)$. This notation means $\mathbf{a} = 3\mathbf{i}+5\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+4\mathbf{j}+7\mathbf{k}$. Using the determinant to find $\mathbf{a}\times\mathbf{b}$ gives

$$\mathbf{a}\times\mathbf{b}=(3\mathbf{i}+5\mathbf{j}-2\mathbf{k})\times(2\mathbf{i}+4\mathbf{j}+7\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&5&-2\\2&4&7\end{pmatrix}$$

$$=\mathbf{i}\det\begin{pmatrix}5&-2\\4&7\end{pmatrix}-\mathbf{j}\det\begin{pmatrix}3&-2\\2&7\end{pmatrix}+\mathbf{k}\det\begin{pmatrix}3&5\\2&4\end{pmatrix}$$

$$=\mathbf{i}(35+8)-\mathbf{j}(21+4)+\mathbf{k}(12-10)$$

$$=43\mathbf{i}-25\mathbf{j}+2\mathbf{k}$$

4. We need to evaluate $\mathbf{B}=\frac{\mu q}{4\pi}\cdot\frac{\mathbf{v}\times\mathbf{r}}{|\mathbf{r}|^2}$ given $\mathbf{r}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{v}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$. The vector product $\mathbf{v}\times\mathbf{r}$ can be found by using (12.24):

$$\mathbf{v}\times\mathbf{r}=(\mathbf{i}-2\mathbf{j}+3\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-2&3\\3&1&-2\end{pmatrix}$$

$$=\mathbf{i}\det\begin{pmatrix}-2&3\\1&-2\end{pmatrix}-\mathbf{j}\det\begin{pmatrix}1&3\\3&-2\end{pmatrix}+\mathbf{k}\det\begin{pmatrix}1&-2\\3&1\end{pmatrix}$$

$$=\mathbf{i}(4-3)-\mathbf{j}(-2-9)+\mathbf{k}(1+6)$$

$$=\mathbf{i}+11\mathbf{j}+7\mathbf{k}$$

The modulus squared $|\mathbf{r}|^2$ is evaluated by

$$|\mathbf{r}|^2=|3\mathbf{i}+\mathbf{j}-2\mathbf{k}|^2=3^2+1^2+(-2)^2=14$$

Substituting these evaluations $\mathbf{v}\times\mathbf{r}=\mathbf{i}+11\mathbf{j}+7\mathbf{k}$ and $|\mathbf{r}|^2=14$ into $\mathbf{B}=\frac{\mu q}{4\pi}\cdot\frac{\mathbf{v}\times\mathbf{r}}{|\mathbf{r}|^2}$:

$$\mathbf{B}=\frac{\mu q}{4\pi}\cdot\frac{\mathbf{v}\times\mathbf{r}}{|\mathbf{r}|^2}=\frac{\mu q}{4\pi}\cdot\frac{\mathbf{i}+11\mathbf{j}+7\mathbf{k}}{14}=\frac{(\mathbf{i}+11\mathbf{j}+7\mathbf{k})\mu q}{56\pi}$$

5. We work with the following vectors $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{c}=-\mathbf{i}+2\mathbf{j}-\mathbf{k}$:

(i) Using the determinant definition to evaluate $\mathbf{a}\times\mathbf{c}$ we have

$$\mathbf{a}\times\mathbf{c}=(2\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+2\mathbf{j}-\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&1\\-1&2&-1\end{pmatrix}$$

$$=\mathbf{i}\det\begin{pmatrix}-1&1\\2&-1\end{pmatrix}-\mathbf{j}\det\begin{pmatrix}2&1\\-1&-1\end{pmatrix}+\mathbf{k}\det\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$$

$$=\mathbf{i}(1-2)-\mathbf{j}(-2+1)+\mathbf{k}(4-1)$$

$$=-\mathbf{i}+\mathbf{j}+3\mathbf{k}$$

(ii) We have found $\mathbf{a}\times\mathbf{c}=-\mathbf{i}+\mathbf{j}+3\mathbf{k}$ in part (i), but what is $\mathbf{c}\times\mathbf{a}$ equal to?

$$\mathbf{c}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{c})=-(-\mathbf{i}+\mathbf{j}+3\mathbf{k})=\mathbf{i}-\mathbf{j}-3\mathbf{k}$$

What does the notation $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})$ mean?

Means the dot product of $\mathbf{a}$ and $\mathbf{c}\times\mathbf{a}$. Thus

$$\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) = (2\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}-3\mathbf{k})$$
$$= 2+1-3 = 0$$

(iii) Using the result of part (i) $\mathbf{a}\times\mathbf{c} = -\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ we have

$$(\mathbf{a}\times\mathbf{c})\times\mathbf{b} = (-\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$$
$$= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 3 \\ 2 & 2 & -1 \end{pmatrix}$$
$$= \mathbf{i}\det\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} - \mathbf{j}\det\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} + \mathbf{k}\det\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$$
$$= \mathbf{i}(-1-6)-\mathbf{j}(1-6)+\mathbf{k}(-2-2) = -7\mathbf{i}+5\mathbf{j}-4\mathbf{k}$$

(iv) Let $\mathbf{r} = (x, 1, 0)$ which means $\mathbf{r} = x\mathbf{i}+\mathbf{j}$. The angle 60° between $\mathbf{b} = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{r} = x\mathbf{i}+\mathbf{j}$ is given by

$$\cos(60^\circ) = \frac{\mathbf{r}\cdot\mathbf{b}}{|\mathbf{r}||\mathbf{b}|}$$
$$= \frac{(x\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k})}{|x\mathbf{i}+\mathbf{j}||2\mathbf{i}+2\mathbf{j}-\mathbf{k}|}$$
$$= \frac{2x+2}{\sqrt{x^2+1}\sqrt{2^2+2^2+(-1)^2}} = \frac{2x+2}{3\sqrt{x^2+1}} \qquad (*)$$

What is $\cos(60^\circ)$ equal to?

$\cos(60^\circ) = \frac{1}{2}$. Substituting this into (*) gives

$$\frac{2x+2}{3\sqrt{x^2+1}} = \frac{1}{2}$$
$$2(2x+2) = 3\sqrt{x^2+1}$$
$$4(2x+2)^2 = 9(x^2+1) \qquad \text{[Squaring both sides]}$$

Expanding both sides we have

$$4(4x^2+8x+4) = 9(x^2+1)$$
$$16x^2+32x+16 = 9x^2+9$$
$$7x^2+32x+7 = 0$$

How do we solve this quadratic equation?

Use the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Using this with $a=7,\ b=32,\ c=7$:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-32\pm\sqrt{32^2-(4\times7\times7)}}{2\times7}$$
$$= \frac{-32\pm\sqrt{828}}{14} = -0.2304,\ -4.3411$$

Substituting $x = -0.2304$ into $\dfrac{2x+2}{3\sqrt{x^2+1}}$ gives $\dfrac{1}{2}$. Substituting the other value of $x = -4.3412$ into $\dfrac{2x+2}{3\sqrt{x^2+1}}$ gives $-\dfrac{1}{2}$. Hence our x value is -0.2304.

6. We work with the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} - \mathbf{j} - 5\mathbf{k}$.

(i) Substituting $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ into

$$\begin{aligned}\mathbf{a} - 2\mathbf{b} + \mathbf{c} &= (\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - 2(2\mathbf{i} + \mathbf{k}) + (3\mathbf{i} - \mathbf{j} - 5\mathbf{k}) \\ &= (1 - 4 + 3)\mathbf{i} + (2 - 1)\mathbf{j} + (-4 - 2 - 5)\mathbf{k} = \mathbf{j} - 11\mathbf{k}\end{aligned}$$

(ii) The dot product $\mathbf{b} \cdot \mathbf{c}$ is found by substituting the given vectors:

$$\mathbf{b} \cdot \mathbf{c} = (2\mathbf{i} + \mathbf{k}) \cdot (3\mathbf{i} - \mathbf{j} - 5\mathbf{k}) = 6 + 0 - 5 = 1$$

(iii) The vector product $\mathbf{b} \times \mathbf{c}$ is given by

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= (2\mathbf{i} + \mathbf{k}) \times (3\mathbf{i} - \mathbf{j} - 5\mathbf{k}) \\ &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 3 & -1 & -5 \end{pmatrix} \\ &= \mathbf{i}\det\begin{pmatrix} 0 & 1 \\ -1 & -5 \end{pmatrix} - \mathbf{j}\det\begin{pmatrix} 2 & 1 \\ 3 & -5 \end{pmatrix} + \mathbf{k}\det\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \\ &= \mathbf{i}(0 + 1) - \mathbf{j}(-10 - 3) + \mathbf{k}(-2 - 0) = \mathbf{i} + 13\mathbf{j} - 2\mathbf{k}\end{aligned}$$

(iv) Need to find $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$. In part (iii) above we found $\mathbf{b} \times \mathbf{c} = \mathbf{i} + 13\mathbf{j} - 2\mathbf{k}$ which means that $\mathbf{c} \times \mathbf{b} = -(\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}) = -\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$. We have to find $\mathbf{a} \times \mathbf{c}$ with $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - \mathbf{j} - 5\mathbf{k}$:

$$\begin{aligned}\mathbf{a} \times \mathbf{c} &= (\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \times (3\mathbf{i} - \mathbf{j} - 5\mathbf{k}) \\ &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -4 \\ 3 & -1 & -5 \end{pmatrix} \\ &= \mathbf{i}\det\begin{pmatrix} 2 & -4 \\ -1 & -5 \end{pmatrix} - \mathbf{j}\det\begin{pmatrix} 1 & -4 \\ 3 & -5 \end{pmatrix} + \mathbf{k}\det\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \\ &= \mathbf{i}(-10 - 4) - \mathbf{j}(-5 + 12) + \mathbf{k}(-1 - 6) = -14\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}\end{aligned}$$

Substituting $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{a} \times \mathbf{c} = -14\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$ into the first term of $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$:

$$\begin{aligned}\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) &= (2\mathbf{i} + \mathbf{k}) \cdot (-14\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}) \\ &= [2 \times (-14)] + [0 \times (-7)] + [1 \times (-7)] = -35\end{aligned}$$

Similarly we have

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) &= (\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \cdot (-\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}) \\ &= -1 - 26 - 8 = -35\end{aligned}$$

Substituting $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -35$ and $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -35$ into $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$ yields

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -35 - (-35) = 0$$

(v) What does the notation $|\mathbf{r}|$ mean?

The modulus of the vector **r**. We need to determine $|\mathbf{c}|-|\mathbf{a}|$:

$$\begin{aligned}|\mathbf{c}|-|\mathbf{a}| &= |3\mathbf{i}-\mathbf{j}-5\mathbf{k}|-|\mathbf{i}+2\mathbf{j}-4\mathbf{k}| \\ &= \sqrt{3^2+(-1)^2+(-5)^2}-\sqrt{1^2+2^2+(-4)^2} \\ &= \sqrt{35}-\sqrt{21}=1.3335\end{aligned}$$

(vi) We are given $\mathbf{b}=2\mathbf{i}+\mathbf{k}$ therefore the unit vector in the direction of **b**, denoted $\hat{b}$, is

$$\hat{b}=\frac{1}{\sqrt{2^2+1^2}}(2\mathbf{i}+\mathbf{k})=\frac{1}{\sqrt{5}}(2\mathbf{i}+\mathbf{k})=\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{k}$$

(vii) Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the vector which is perpendicular to vectors **a** and **b**. We have $\mathbf{r}\cdot\mathbf{a}=0$ and $\mathbf{r}\cdot\mathbf{b}=0$. From $\mathbf{r}\cdot\mathbf{a}=0$ we have

$$\begin{aligned}\mathbf{r}\cdot\mathbf{a} &= (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-4\mathbf{k}) \\ &= x+2y-4z=0 \qquad (\dagger)\end{aligned}$$

Using $\mathbf{r}\cdot\mathbf{b}=0$ yields

$$\begin{aligned}\mathbf{r}\cdot\mathbf{b} &= (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(2\mathbf{i}+\mathbf{k}) \\ &= 2x+z=0 \quad\Rightarrow\quad z=-2x\end{aligned}$$

Let $x=1$ then $z=-2$. Substituting these values into $(\dagger)$ gives

$$1+2y-4(-2)=0 \quad\Rightarrow\quad y=-\frac{9}{2}=-4.5$$

Hence $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\mathbf{i}-4.5\mathbf{j}-2\mathbf{k}$. Finding the unit vector $\hat{\mathbf{r}}$ we have

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{1}{\sqrt{1^2+(-4.5)^2+(-2)^2}}(\mathbf{i}-4.5\mathbf{j}-2\mathbf{k}) \\ &= \frac{1}{\sqrt{25.25}}(\mathbf{i}-4.5\mathbf{j}-2\mathbf{k})\end{aligned}$$

7. We can write the given vectors $\mathbf{u}=(1,-1,1)$, $\mathbf{v}=\geq(0,1,2)$, $\mathbf{w}=(1,0,5)$ in terms of **i**, **j** and **k** which means $\mathbf{u}=\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{v}=\mathbf{j}+2\mathbf{k}$, $\mathbf{w}=\mathbf{i}+5\mathbf{k}$. The first vector product is

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (\mathbf{i}-\mathbf{j}+\mathbf{k})\times(\mathbf{j}+2\mathbf{k}) \\ &= \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 1 & 2\end{pmatrix} \\ &= \mathbf{i}\det\begin{pmatrix}-1 & 1 \\ 1 & 2\end{pmatrix}-\mathbf{j}\det\begin{pmatrix}1 & 1 \\ 0 & 2\end{pmatrix}+\mathbf{k}\det\begin{pmatrix}1 & -1 \\ 0 & 1\end{pmatrix} \\ &= \mathbf{i}(-2-1)-\mathbf{j}(2-0)+\mathbf{k}(1-0)=-3\mathbf{i}-2\mathbf{j}+\mathbf{k}\end{aligned}$$

The cosine of the angle, say θ, between the vectors $\mathbf{u}\times\mathbf{v}$ and **w** is given by

$$\cos(\theta)=\frac{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}}{|\mathbf{u}\times\mathbf{v}||\mathbf{w}|} \qquad (*)$$

Determining the numerator and denominator of (*):

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(-3\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+5\mathbf{k})=-3+5=2$$

$$
\begin{aligned}
|\mathbf{u}\times\mathbf{v}||\mathbf{w}| &= |-3\mathbf{i}-2\mathbf{j}+\mathbf{k}||\mathbf{i}+5\mathbf{k}| \\
&= \sqrt{(-3)^2+(-2)^2+1^2}\sqrt{1^2+5^2} \\
&= \sqrt{14}\sqrt{26}
\end{aligned}
$$

Putting these results into (*) gives

$$
\cos(\theta) = \frac{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}}{|\mathbf{u}\times\mathbf{v}||\mathbf{w}|} = \frac{2}{\sqrt{26}\sqrt{14}}
$$

$$
= \frac{2}{\sqrt{26\times 14}} = \frac{2}{\sqrt{4\times 91}} = \frac{\cancel{2}}{\cancel{2}\sqrt{91}} = \frac{1}{\sqrt{91}} \qquad [\text{Cancelling 2's}]
$$

Similarly we can find the cosine of the angle between $\mathbf{u}\times\mathbf{w}$ and $\mathbf{v}$:

$$
\begin{aligned}
\mathbf{u}\times\mathbf{w} &= (\mathbf{i}-\mathbf{j}+\mathbf{k})\times(\mathbf{i}+5\mathbf{k}) \\
&= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 0 & 5 \end{pmatrix} \\
&= \mathbf{i}\det\begin{pmatrix} -1 & 1 \\ 0 & 5 \end{pmatrix} - \mathbf{j}\det\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} + \mathbf{k}\det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \mathbf{i}(-5-0)-\mathbf{j}(5-1)+\mathbf{k}(0-(-1)) = -5\mathbf{i}-4\mathbf{j}+\mathbf{k}
\end{aligned}
$$

Need to determine $(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ and $|\mathbf{u}\times\mathbf{w}|$, $|\mathbf{v}|$:

$$
(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v} = (-5\mathbf{i}-4\mathbf{j}+\mathbf{k})\cdot(\mathbf{j}+2\mathbf{k}) = 0-4+2 = -2
$$

$$
|\mathbf{u}\times\mathbf{w}| = |-5\mathbf{i}-4\mathbf{j}+\mathbf{k}| = \sqrt{(-5)^2+(-4)^2+1^2} = \sqrt{42}
$$

$$
|\mathbf{v}| = |\mathbf{j}+2\mathbf{k}| = \sqrt{1^2+2^2} = \sqrt{5}
$$

Substituting $(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v} = -2$ and $|\mathbf{u}\times\mathbf{w}| = \sqrt{42}$, $|\mathbf{v}| = \sqrt{5}$ into $\cos(\theta) = \dfrac{(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}}{|\mathbf{u}\times\mathbf{w}||\mathbf{v}|}$ yields

$$
\cos(\theta) = \frac{(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}}{|\mathbf{u}\times\mathbf{w}||\mathbf{v}|} = -\frac{2}{\sqrt{42}\sqrt{5}} = -\frac{2}{\sqrt{210}}
$$