

Complete Solutions to Examination Questions 13

1. (i) Separating the variables of the given differential equation $y \frac{dy}{dx} = 3x^2$:

$$\int y dy = \int 3x^2 dx$$

$$\frac{y^2}{2} = x^3 + C \quad \text{[Integrating]}$$

$$y^2 = 2x^3 + A \quad \text{where } A = 2C$$

$$y = \sqrt{2x^3 + A} \quad \text{[Taking square root]}$$

Using the given initial condition $y(0) = 1$ which means that when $x = 0$, $y = 1$:

$$\sqrt{2(0)^3 + A} = 1 \Rightarrow A = 1$$

Our particular solution is $y = \sqrt{2x^3 + 1}$.

- (ii) We use integrating factor method to solve the given initial value problem

$$\frac{dy}{dx} - y = e^{2x} \quad y(0) = 0$$

$IF = e^{\int -1 dx} = e^{-x}$. Using the IF formula (13.4) we have

$$y(IF) = \int e^{2x} (IF) dx$$

$$ye^{-x} = \int e^{2x} e^{-x} dx$$

$$\stackrel{\text{using } a^m a^n = a^{m+n}}{=} \int e^{2x-x} dx = \int e^x dx \stackrel{\text{Integrating}}{=} e^x + C$$

Multiplying through by e^x gives

$$y = e^x e^x + Ce^x = e^{2x} + Ce^x \quad (\odot)$$

Applying the given initial condition $y(0) = 0$ which means $x = 0$, $y = 0$:

$$e^{2 \times 0} + Ce^0 = 1 + C = 0 \Rightarrow C = -1$$

Substituting this $C = -1$ into (\odot) gives our particular solution $y = e^{2x} - e^x$.

2. (a) Divide the given differential equation by 10 so that the numbers are slightly easier to deal with:

$$15 \frac{dv}{dt} = 35 - 1.5v$$

Separating variables yields

$$\frac{15}{35 - 1.5v} dv = dt$$

Differentiating the denominator gives -1.5 therefore integrating the left hand side:

$$\int \frac{15}{35 - 1.5v} dv = \frac{15}{-1.5} \int \frac{-1.5}{35 - 1.5v} dv$$

$$= -10 \int \frac{-1.5}{35 - 1.5v} dv$$

$$= -10 \ln|35 - 1.5v| \quad \left[\text{Using } \int \frac{f'}{f} dx = \ln|f| \right]$$

We can add the constant to the right hand side. What is $\int dt$ equal to?

$t + C$. We have

$$-10 \ln|35 - 1.5v| = t + C \quad (\dagger)$$

Since we are given initial condition $(t = 0, v = 0)$ we can determine C . Substituting these into the above

$$-10 \ln|35 - 1.5(0)| = 0 + C \Rightarrow C = -10 \ln(35)$$

Substituting $C = -10 \ln(35)$ into (\dagger) gives

$$-10 \ln|35 - 1.5v| = t - 10 \ln(35) \quad (\ddagger)$$

We need to pull out v . How?

Divide through by -10 and then take exponential of both sides:

$$\ln|35 - 1.5v| = -\frac{t}{10} + \ln(35)$$

$$35 - 1.5v = e^{-\frac{t}{10} + \ln(35)} = e^{-\frac{t}{10}} e^{\ln(35)} = 35e^{-\frac{t}{10}}$$

We have $35 - 1.5v = 35e^{-\frac{t}{10}}$. Rearranging:

$$35 - 35e^{-\frac{t}{10}} = 1.5v \text{ implies that } v = \frac{35 - 35e^{-\frac{t}{10}}}{1.5} = \frac{35}{1.5} \left(1 - e^{-\frac{t}{10}}\right)$$

The expression for velocity is $v = \frac{35}{1.5} \left(1 - e^{-\frac{t}{10}}\right)$.

(b) Putting $v = 2$ into (\ddagger) :

$$t = -10 \ln|35 - 1.5(2)| + 10 \ln(35) = 0.8961$$

3. The differential equation is given as $\frac{dT}{dt} = -k(T - 70)$. Separating variables

$$\frac{dT}{T - 70} = -k dt$$

Integrating both sides yields

$$\ln(T - 70) = -kt + C \quad (*)$$

Substituting the given initial condition $t = 0, T = 200$ into $(*)$ gives

$$\ln(200 - 70) = 0 + C \Rightarrow C = \ln(130) = 4.8675$$

Substituting the other condition $t = 15, T = 120$ and $C = 4.8675$ into $(*)$:

$$\ln(120 - 70) = -15k + 4.8675$$

Transposing to make k the subject

$$\frac{\ln(120 - 70) - 4.8675}{-15} = k \Rightarrow k = 0.0637$$

Substituting $k = 0.0637$ and $C = 4.8675$ values into $(*)$ gives

$$\ln(T - 70) = -0.0637t + 4.8675$$

We need to find the time t when $T = 90$:

$$\ln(90 - 70) = 2.9957 = -0.0637t + 4.8675$$

Transposing to make t the subject results in

$$t = \frac{2.9957 - 4.8675}{-0.0637} = 29.38$$

Dr Gourley needs to wait 29.38 minutes or 29 minutes and 23 seconds from 10.15 a.m. This means Dr Gourley starts to drink his coffee at 10.44 a.m. (to the nearest minute).

4. (i) The integrating factor (IF) is given by

$$IF = \exp\left(\int \frac{1}{x} dx\right) = \exp[\ln(x)] = x \quad [\text{Remember } \exp[\ln(a)] = a]$$

Using the formula we have

$$\begin{aligned} y(IF) &= \int x(IF) dx \\ yx &= \int xx dx = \int x^2 dx \\ &= \frac{x^3}{3} + C \quad [\text{Integrating}] \end{aligned}$$

Divide through by x to find y :

$$y = \frac{x^3}{3x} + \frac{C}{x} = \frac{x^2}{3} + \frac{C}{x} \quad (\dagger)$$

Substituting the given initial condition $y = 3$ at $x = 1$:

$$\frac{1^2}{3} + \frac{C}{1} = \frac{1}{3} + C = 3 \quad \text{implies} \quad C = \frac{8}{3}$$

The particular solution is found by substituting $C = \frac{8}{3}$ into (\dagger) :

$$y = \frac{x^2}{3} + \frac{8}{3x} = \frac{1}{3} \left(x^2 + \frac{8}{x} \right) \quad \left[\text{Taking out } \frac{1}{3} \right]$$

(ii) We are instructed to use the substitution $y = vx$. Differentiating this $y = vx$ by the

product rule gives $\frac{dy}{dx} = \frac{dv}{dx}x + v$. Thus the given differential equation $\frac{dy}{dx} = \frac{y}{x-y}$ becomes

$$\frac{dv}{dx}x + v = \frac{vx}{x-vx} = \frac{v}{1-v} \quad [\text{Cancelling } x\text{'s}]$$

$$\begin{aligned} \frac{dv}{dx}x &= \frac{v}{1-v} - v \\ &= \frac{v - v(1-v)}{1-v} = \frac{v^2}{1-v} \end{aligned}$$

We can now separate the variables of $\frac{dv}{dx}x = \frac{v^2}{1-v}$:

$$\frac{1-v}{v^2} dv = \frac{dx}{x}$$

Integrating both sides:

$$\begin{aligned} \int \left(\frac{1-v}{v^2} \right) dv &= \int \frac{dx}{x} \\ \int \left(v^{-2} - \frac{1}{v} \right) dv &= \ln|x| + C \\ -v^{-1} - \ln|v| &= \ln|x| + C \end{aligned}$$

From the given substitution $y = vx$ we have $v = \frac{y}{x}$ therefore $v^{-1} = \frac{x}{y}$. Substituting these into the above $-v^{-1} - \ln|v| = \ln|x| + C$ or $-v^{-1} = \ln|v| + \ln|x| + C$

$$-\frac{x}{y} = \ln\left|\frac{y}{x}\right| + \ln|x| + C = \underbrace{\ln|y| - \ln|x|}_{\text{Using } \ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)} + \ln|x| + C = \ln|y| + C$$

Multiplying through by y :

$$-x = y[\ln|y| + C]$$

5. We need to use the rules of indices on $y' = e^{x+2y}$. We have

$$\frac{dy}{dx} = e^x e^{2y} \quad \left[\text{Applying } a^{m+n} = a^m a^n \right]$$

$$\frac{dy}{e^{2y}} = e^{-2y} dy = e^x dx \quad \left[\text{Separating variables} \right]$$

Integrating both sides

$$\begin{aligned} \int e^{-2y} dy &= \int e^x dx \\ \frac{e^{-2y}}{-2} &= e^x + C \\ e^{-2y} &= -2(e^x + C) \end{aligned}$$

Taking logs of both sides yields

$$\begin{aligned} \ln(e^{-2y}) &= \ln|-2(e^x + C)| \\ -2y &= \ln|-2(e^x + C)| \quad \text{implies that } y = -\frac{1}{2} \ln|-2(e^x + C)| \end{aligned}$$

Hence our solution is $y = -\frac{1}{2} \ln|-2(e^x + C)|$.

6. Let $f'(x) = \frac{dy}{dx}$ then

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x + \sqrt{x}}{x} \\ &= \frac{2x}{x} + \frac{\sqrt{x}}{x} = 2 + \frac{x^{1/2}}{x} = 2 + x^{-1/2} \quad \left[\text{Because } \frac{x^{1/2}}{x} = x^{1/2-1} = x^{-1/2} \right] \end{aligned}$$

Separating variables we have $dy = (2 + x^{-1/2}) dx$. Integrating this

$$\begin{aligned} \int dy &= \int (2 + x^{-1/2}) dx \\ y &= 2x + \frac{x^{1/2}}{1/2} + C = 2x + 2x^{1/2} + C \quad (*) \end{aligned}$$

We can find a particular solution in this case because we have been given the initial condition $f(1) = y(1) = 3$. What does this mean?

When $x = 1$ then $y = 3$. Substituting these into (*) gives

$$2(1) + 2(1^{1/2}) + C = 3 \quad \text{implies that } C = -1$$

Putting $C = -1$ into (*) gives the particular solution $y = 2x + 2x^{1/2} - 1$.

7. We need to use Euler's method which is

$$(13.12) \quad y_{n+1} = y_n + h[f(x_n, y_n)]$$

In this case we have $f(x_n, y_n) = y_n - 2x_n$ and $h = 0.5$. Substituting these into the above formula (13.12):

$$y_{n+1} = y_n + 0.5[y_n - 2x_n] \quad (\dagger)$$

Step 1

Starting with $n = 0$ gives

$$y_1 = y_0 + 0.5[y_0 - 2x_0] \quad (*)$$

What are the values of x_0 and y_0 ?

These are given by the initial condition in the question, $y(1) = 0$, which means that $x_0 = 1$, $y_0 = 0$. Substituting this into (*) yields

$$y_1 = 0 + 0.5[0 - 2(1)] = -1$$

The y value at $x_1 = 1.5$ is $y_1 = -1$.

Step 2

How do we calculate y_2 ?

By substituting $n = 1$ into (\dagger):

$$y_2 = y_1 + 0.5[y_1 - 2x_1]$$

We have evaluated $x_1 = 1.5$ and $y_1 = -1$ in the first step. Therefore

$$y_2 = -1 + 0.5[-1 - 2(1.5)] = -3$$

Repeating this process we have the following:

Step 3

We use the above evaluations $x_2 = 2$ and $y_2 = -3$ with $n = 2$ into (\dagger):

$$y_3 = y_2 + 0.5[y_2 - 2x_2]$$

$$y_3 = -3 + 0.5[-3 - 2(2)] = -6.5$$

Step 4

Similarly for $n = 3$ with $x_3 = 2.5$ and $y_3 = -6.5$ in (\dagger) we have

$$\begin{aligned} y_4 &= y_3 + 0.5[y_3 - 2x_3] \\ &= -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25 \end{aligned}$$

Thus at $x = 3$ we have $y_4 = -12.25$. This means that $y(3) = -12.25$.

8. We need to differentiate $y = \ln(\cos x)$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cos(x)}[-\sin(x)] \\ &= -\frac{\sin(x)}{\cos(x)} = -\tan(x) \quad \left[\text{Because } \frac{\sin}{\cos} = \tan \right] \end{aligned}$$

The integrating factor IF of $\frac{dy}{dx} + 2y \tan x = \sin x$ is

$$\begin{aligned} \exp\left(\int 2 \tan(x) dx\right) &= \exp\left\{\int -2[-\tan(x)] dx\right\} \\ &= \exp\left[-2\int -\tan(x) dx\right] \quad [\text{Taking } -2 \text{ out of integral}] \\ &= \exp\left\{-2\ln[\cos(x)]\right\} \quad [\text{By the above result}] \\ &= \exp\left\{\ln[\cos(x)^{-2}]\right\} \quad [\text{Because } n \ln(a) = \ln(a^n)] \\ &= [\cos(x)]^{-2} = \frac{1}{\cos^2(x)} \quad [\text{Because } \exp[\ln(a)] = a] \end{aligned}$$

IF = $\frac{1}{\cos^2(x)}$. Using the formula for integrating factor we have

$$y(IF) = \int \sin(x)(IF) dx$$

Substituting IF = $\frac{1}{\cos^2(x)}$ into this yields

$$y \frac{1}{\cos^2(x)} = \int \sin(x) \frac{1}{\cos^2(x)} dx = \int \frac{\sin(x)}{\cos^2(x)} dx \quad (*)$$

How do we find $\int \frac{\sin(x)}{\cos^2(x)} dx$?

Using integration by substitution. Let $u = \cos(x)$ then

$$\frac{du}{dx} = -\sin(x) \quad \text{implies that} \quad dx = -\frac{du}{\sin(x)}$$

Putting these into $\int \frac{\sin(x)}{\cos^2(x)} dx$:

$$\begin{aligned} \int \frac{\sin(x)}{\cos^2(x)} dx &= \int \frac{-\sin(x)}{u^2} \frac{du}{\sin(x)} \\ &= \int -u^{-2} du = u^{-1} + C = \frac{1}{\cos(x)} + C \quad [\text{Substituting } u = \cos(x)] \end{aligned}$$

Substituting this $\int \frac{\sin(x)}{\cos^2(x)} dx = \frac{1}{\cos(x)} + C$ into the right hand side of (*) gives

$$\begin{aligned} y \frac{1}{\cos^2(x)} &= \frac{1}{\cos(x)} + C \\ y &= \frac{\cos^2(x)}{\cos(x)} + C \cos^2(x) = \cos(x) + C \cos^2(x) \quad (**) \end{aligned}$$

By using the initial condition $y(\pi/3) = 0$ we can find the value of C . What does this notation $y(\pi/3) = 0$ mean?

When $x = \frac{\pi}{3}$ then $y = 0$:

$$\cos(\pi/3) + C \cos^2(\pi/3) = 0$$

$$\frac{1}{2} + \frac{1}{4}C = 0 \quad \text{implies} \quad C = -\frac{1/2}{1/4} = -2$$

Putting $C = -2$ into (***) gives $y = \cos(x) - 2 \cos^2(x)$.

9. How do we solve the given first order differential equation

$$\frac{dy}{dt} - 2 \tan(t) y = \tan(t), \quad y\left(\frac{\pi}{4}\right) = 5?$$

Using integrating factor (IF). IF is given by

$$\begin{aligned} IF &= \exp\left[\int -2 \tan(t) dt\right] = \exp\left[-2 \int \tan(t) dt\right] \quad [\text{Taking } -2 \text{ out of integral}] \\ &= \exp\{-2 \ln[\sec(t)]\} \quad [\text{Because } \int \tan(t) dt = \ln[\sec(t)]] \\ &= [\sec(t)]^{-2} = \frac{1}{\sec^2(t)} = \cos^2(t) \end{aligned}$$

Applying the integrating factor formula with $IF = \cos^2(t)$ we have

$$\begin{aligned} y \cos^2(t) &= \int \tan(t) \cos^2(t) dt \\ &\stackrel{\substack{\equiv \\ \text{Because } \tan = \frac{\sin}{\cos}}}{=} \int \frac{\sin(t)}{\cos(t)} \cos^2(t) dt \quad \stackrel{\substack{\equiv \\ \text{Cancelling } \cos(t)'s}}{=} \int \sin(t) \cos(t) dt \end{aligned}$$

How do we find $\int \sin(t) \cos(t) dt$?

Note that the derivative of \sin is \cos therefore by substitution we have

$$\begin{aligned} \int \sin(t) \cos(t) dt &= \int \sin(t) d[\sin(t)] \\ &= \frac{\sin^2(t)}{2} + C \quad \left[\text{Using } \int u du = \frac{u^2}{2} \right] \end{aligned}$$

By substituting this into the right hand side of the above result $y \cos^2(t) = \int \sin(t) \cos(t) dt$:

$$\begin{aligned} y \cos^2(t) &= \frac{\sin^2(t)}{2} + C \\ y &= \frac{\sin^2(t)}{2 \cos^2(t)} + \frac{C}{\cos^2(t)} \\ &= \frac{\tan^2(t)}{2} + C \sec^2(t) \quad \left[\text{Because } \frac{\sin}{\cos} = \tan \text{ and } \frac{1}{\cos} = \sec \right] \end{aligned}$$

Using the initial condition $y\left(\frac{\pi}{4}\right) = 5$ which means that when $t = \frac{\pi}{4}$ then $y = 5$:

$$\frac{\tan^2(\pi/4)}{2} + C \sec^2\left(\frac{\pi}{4}\right) = 5$$

$$\frac{1}{2} + C(\sqrt{2})^2 = \frac{1}{2} + 2C = 5 \quad \Rightarrow \quad 2C = 5 - \frac{1}{2} = \frac{9}{2}$$

Dividing by 2 gives $C = \frac{9}{4}$. Our solution is

$$\begin{aligned}
 y &= \frac{\tan^2(t)}{2} + \frac{9}{4} \sec^2(t) = \frac{1}{4} [2 \tan^2(t) + 9 \sec^2(t)] \\
 &= \frac{1}{4} \{2 \tan^2(t) + 9 [1 + \tan^2(t)]\} \quad [\text{Because } \sec^2(t) = 1 + \tan^2(t)] \\
 &= \frac{1}{4} [11 \tan^2(t) + 9]
 \end{aligned}$$

10. Dividing the given differential equation by x we have $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin(x)}{x}$. We use the integrating factor method to solve this differential equation. The integrating factor, IF, is

$$IF = e^{\int \frac{2}{x} dx} = e^{2 \ln|x|} = e^{\ln(x^2)} = x^2 \quad [\text{Because } e^{\ln(a)} = a]$$

Using the integrating formula method we have

$$yx^2 = \int x^2 \frac{\sin(x)}{x} dx = \int x \sin(x) dx \quad (\odot)$$

How do we integrate the right hand integrand $x \sin(x)$?

Use integration by parts formula. Let

$$\begin{aligned}
 u &= x & v' &= \sin(x) \\
 u' &= 1 & v &= \int \sin(x) dx = -\cos(x)
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int x \sin(x) dx &= uv - \int u' v dx \\
 &= x[-\cos(x)] - \int (1)[- \cos(x)] dx \\
 &= -x \cos(x) + \int \cos(x) = -x \cos(x) + \sin(x) + C
 \end{aligned}$$

Substituting this $\int x \sin(x) dx = -x \cos(x) + \sin(x) + C$ into the right hand side of (\odot) gives

$$\begin{aligned}
 yx^2 &= -x \cos(x) + \sin(x) + C \\
 y &= \frac{\sin(x) - x \cos(x) + C}{x^2}
 \end{aligned}$$

11. (a) We have the first order differential equation $m' = 16m \left(1 - \frac{m}{4}\right)$. Using the fact stated in the question with $k = 16$ and $K = 4$ we have

$$m = \frac{K}{1 + Ae^{-kt}} = \frac{4}{1 + Ae^{-16t}} \quad (\dagger)$$

We are given the initial condition when $t = 0$, $m = 2$. Substituting this into the above

$$\frac{4}{1 + Ae^{-16 \times 0}} = \frac{4}{1 + A} = 2$$

Transposing this to make A the subject of the formula:

$$1 + A = \frac{4}{2} = 2 \quad \text{gives } A = 2 - 1 = 1$$

Substituting $A = 1$ into (†) yields $m = \frac{4}{1 + e^{-16t}}$. We need to find when the number of fish will equal 3 million. How?

Substitute $m = 3$ and find t from $m = \frac{4}{1 + e^{-16t}}$:

$$\begin{aligned}\frac{4}{1 + e^{-16t}} &= 3 \\ \frac{4}{3} &= 1 + e^{-16t} \Rightarrow \frac{4}{3} - 1 = \frac{1}{3} = e^{-16t}\end{aligned}$$

How do we determine t from $e^{-16t} = \frac{1}{3}$?

Take logs of both sides:

$$\begin{aligned}\ln(e^{-16t}) &= \ln\left(\frac{1}{3}\right) \\ -16t &= \ln(1/3) \Rightarrow t = \frac{\ln(1/3)}{-16} = 0.0687\end{aligned}$$

There will be 3 million fish in the lake after $0.0687 \times 365 = 25$ days. [Assuming it is not a leap year.]

(b) Expanding the given differential equation

$$\begin{aligned}\frac{dm}{dt} &= 16m - 16m\left(\frac{m}{4}\right) - 12 \\ &= 16m - 4m^2 - 12 = -4(m^2 - 4m + 3)\end{aligned}$$

Separating the variables gives

$$\frac{dm}{m^2 - 4m + 3} = -4dt \quad (*)$$

Need to integrate in order to find m . How?

The right hand side integral is straightforward. To integrate the left hand side we factorise the denominator and then use partial fractions.

$$\begin{aligned}\frac{1}{m^2 - 4m + 3} &= \frac{1}{(m-3)(m-1)} \\ &= \frac{A}{m-3} + \frac{B}{m-1}\end{aligned}$$

Using the cover – up rule or cross multiplying we have $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Thus

$$\frac{1}{m^2 - 4m + 3} = \frac{1}{2} \frac{1}{m-3} - \frac{1}{2} \frac{1}{m-1} = \frac{1}{2} \left[\frac{1}{m-3} - \frac{1}{m-1} \right]$$

Integrating this gives

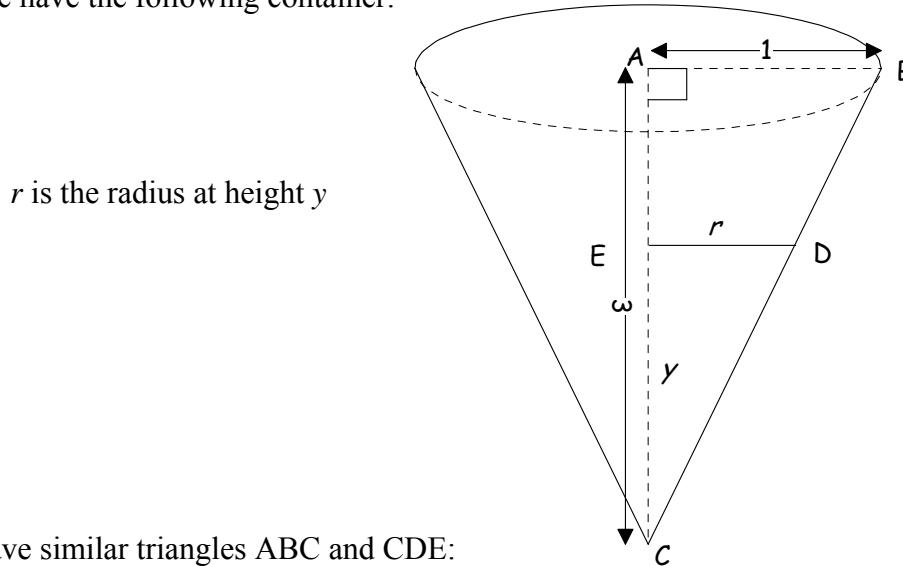
$$\begin{aligned}\int \frac{dm}{m^2 - 4m + 3} &= \frac{1}{2} \left[\int \frac{dm}{m-3} - \int \frac{dm}{m-1} \right] \\ &= \frac{1}{2} [\ln|m-3| - \ln|m-1|] \\ &= \frac{1}{2} \ln \left| \frac{m-3}{m-1} \right| \quad \left[\text{Because } \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right) \right]\end{aligned}$$

Putting this into the above equation of (*) with the integral of -4 being $-4t$:

$$\frac{1}{2} \ln \left| \frac{m-3}{m-1} \right| = -4t + C$$

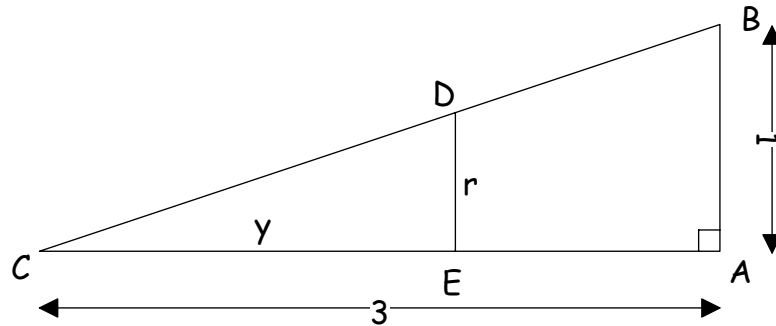
From this equation we know the fish population cannot equal 3 million because substituting $m = 3$ will mean we need to evaluate $\ln(0)$ which is impossible. There is **no** such real t value.

12. We have the following container:



r is the radius at height y

We have similar triangles ABC and CDE:



From properties of similar triangles we have

$$\frac{AB}{AC} = \frac{DE}{CE}$$

Substituting the above values in the diagram results in

$$\frac{1}{3} = \frac{r}{y} \quad \text{implies that} \quad r = \frac{y}{3}$$

The cross-sectional area $A(y)$ is a circle with radius $r = \frac{y}{3}$:

$$A(y) = \pi r^2 = \pi \left(\frac{y}{3} \right)^2 = \frac{\pi y^2}{9}$$

a is the cross-sectional area of the exit hole which is a circle of radius $1\text{cm} = 0.01\text{m}$. Hence

$$a = \pi (0.01)^2 = (1 \times 10^{-4}) \pi$$

Substituting these $A(y) = \frac{\pi y^2}{9}$ and $a = (1 \times 10^{-4})\pi$ into the given Torricelli's formula

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy}$$

gives

$$\frac{\pi y^2}{9} \frac{dy}{dt} = -(1 \times 10^{-4})\pi\sqrt{2gy}$$

Rearranging this we have

$$\frac{dy}{dt} = -\frac{9(1 \times 10^{-4})\pi}{\pi y^2} (2gy)^{1/2} = -\frac{(9 \times 10^{-4})}{y^2} (2g)^{1/2} y^{1/2}$$

We are given that $g = 9.8 \text{ m/sec}^2$ and using the rules of indices on y :

$$\begin{aligned} \frac{dy}{dt} &= -\frac{(9 \times 10^{-4})}{y^{2-1/2}} (2 \times 9.8)^{1/2} \\ &= -\frac{(9 \times 10^{-4})}{y^{3/2}} 4.4272 = -\frac{4 \times 10^{-3}}{y^{3/2}} \end{aligned}$$

Separating the variables of this differential equation $\frac{dy}{dt} = -\frac{4 \times 10^{-3}}{y^{3/2}}$ yields

$$\int y^{3/2} dy = -\int 4 \times 10^{-3} dt$$

$$\frac{2y^{5/2}}{5} = -(4 \times 10^{-3})t + C$$

$$y^{5/2} = -\frac{5}{2}(4 \times 10^{-3})t + \frac{5}{2}C = -0.01t + A \quad \text{where } A = \frac{5}{2}C$$

Substituting the initial condition when $t = 0$, $y = 3$ into this $y^{5/2} = -0.01t + A$

$$3^{5/2} = 0 + A \Rightarrow A = 3^{5/2} = 15.589$$

Our solution is $y^{5/2} = -0.01t + 15.589$ implies $y = (-0.01t + 15.589)^{2/5}$.

The tank will be empty when $y = 0$. Substituting $y = 0$ into $(-0.01t + 15.589)^{2/5} = y$:

$$-0.01t + 15.589 = 0 \Rightarrow t = \frac{15.589}{0.01} = 1558.9$$

It will take 1558.9 seconds or $\frac{1558.9}{60} = 25.98$ minutes to empty.