Complete Solutions to Examination Questions 14

1. The characteristic equation is

$$m^2 + 2m + 2 = 0$$

Solving this quadratic equation by the formula method $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1,\ b = 2$ and $c = 2$ gives

$$m = \frac{-2 \pm \sqrt{2^2 - (4 \times 1 \times 2)}}{2 \times 1} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2}{2} \pm j\frac{2}{2} = -1 \pm j$$

The general solution is given by

$$y = e^{-x}\left[A\cos(x) + B\sin(x)\right] \qquad (*)$$

Substituting the given initial condition $y = 1$ at $x = 0$ into (*):

$$\underbrace{e^0}_{=1}\left[A\underbrace{\cos(0)}_{=1} + B\underbrace{\sin(0)}_{=0}\right] = A = 1$$

To use the other initial condition we need to differentiate (*):

$$y = e^{-x}\left[A\cos(x) + B\sin(x)\right]$$

$$\frac{dy}{dx} \underbrace{=}_{\text{By Product Rule}} -e^{-x}\left[A\cos(x) + B\sin(x)\right] + e^{-x}\left[-A\sin(x) + B\cos(x)\right]$$

$$= e^{-x}\left[-A\cos(x) - B\sin(x) - A\sin(x) + B\cos(x)\right]$$

Substituting the other initial condition $\frac{dy}{dx} = 0$ at $x = 0$ into this result:

$$\underbrace{e^0}_{=1}\left[-A\underbrace{\cos(0)}_{=1} - B\underbrace{\sin(0)}_{=0} - A\underbrace{\sin(0)}_{=0} + B\underbrace{\cos(0)}_{=1}\right] = -A + B = 0$$

From above we have $A = 1$ therefore $B = 1$. Our general solution is found by substituting these values $A = 1$ and $B = 1$ into (*):

$$y = e^{-x}\left[\cos(x) + \sin(x)\right]$$

2. (i) The characteristic equation is $m^2 + 16 = 0$. Solving this gives

$$m^2 = -16$$

$$m = \sqrt{-16} = \pm j4 = 0 \pm j4$$

Since we have complex roots

$$y = A\cos(4x) + B\sin(4x) \qquad (*)$$

We are given the initial conditions $y(0) = 3$ and $y'(0) = -2$. Substituting the first of these conditions $y(0) = 3$ which means that when $x = 0$, $y = 3$ into (*):

$$A\underbrace{\cos(4 \times 0)}_{=1} + B\underbrace{\sin(4 \times 0)}_{=0} = 3 \quad \text{gives } A = 3$$

Next we apply the second condition $y'(0) = -2$ which means that when $x = 0$, $y' = -2$.
Differentiating (*) gives

$$y' = -4A\sin(4x) + 4B\cos(4x) \qquad \left[\begin{array}{l}\text{Because } \left[\cos(kx)\right]' = -k\sin(kx) \\ \text{and } \left[\sin(kx)\right]' = k\cos(kx)\end{array}\right]$$

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Substituting $x = 0$, $y' = -2$ into the above yields

$$-4A\underbrace{\sin(4\times 0)}_{=0} + 4B\underbrace{\cos(4\times 0)}_{=1} = -2$$

$$4B = -2 \quad \Rightarrow \quad B = -\frac{2}{4} = -\frac{1}{2}$$

Our particular solution is found by substituting $A = 3$ and $B = -\frac{1}{2}$ into (*):

$$y = A\cos(4x) + B\sin(4x) = 3\cos(4x) - \frac{1}{2}\sin(4x)$$

(ii) For solving the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 2e^{4x}$ we first find the homogeneous solution:

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$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

The characteristic equation is

$$m^2 + 2m - 15 = 0$$

$$(m-3)(m+5) = 0 \qquad [\text{Factorizing}]$$

$$m_1 = 3, \quad m_2 = -5$$

Our complementary function is $y_c = Ae^{3x} + Be^{-5x}$. What is the trail function for the particular integral in this case?

$Y = Ce^{4x}$. Differentiating this gives

$$Y' = 4Ce^{4x} \qquad \left[\text{Because } \left(e^{kx}\right)' = ke^{kx}\right]$$

$$Y'' = 16Ce^{4x} \qquad \left[\text{Because } \left(e^{kx}\right)' = ke^{kx}\right]$$

Substituting these into the given differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 2e^{4x}$ yields

$$16Ce^{4x} + 2\left(4Ce^{4x}\right) - 15Ce^{4x} = 2e^{4x}$$

$$(16C + 8C - 15C)e^{4x} = 2e^{4x}$$

$$9C = 2 \text{ implies that } C = \frac{2}{9}$$

Hence the particular integral is $Y = Ce^{4x} = \frac{2}{9}e^{4x}$.

Our general solution to the differential equation is

$$y = y_c + Y = Ae^{3x} + Be^{-5x} + \frac{2}{9}e^{4x}$$

3. (a) We need to solve the given differential equation $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + 25s = 0$. The characteristic equation is

$$m^2 + 25 = 0 \quad \text{gives} \quad m = \sqrt{-25} = \pm j5$$

The general solution is

$$s = A\cos(5t) + B\sin(5t) \qquad (☼)$$

Substituting the given initial condition when $t = 0$, $s = 2$ we have

$$A\underbrace{\cos(5\times 0)}_{=1} + B\underbrace{\sin(5\times 0)}_{=0} = 2 \quad \text{gives} \quad A = 2$$

The other initial condition is $\frac{\mathrm{d}s}{\mathrm{d}t} = 5$ when $t = 0$. Differentiating (☼) yields

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$$\frac{\mathrm{d}s}{\mathrm{d}t} = -5A\sin(5t) + 5B\cos(5t) \qquad \left[\begin{array}{l}\text{Because } [\cos(kt)]' = -k\sin(kt) \\ \text{and } [\sin(kt)]' = k\cos(kt)\end{array}\right]$$

Substituting the other initial condition:

$$-5A\underbrace{\sin(5\times 0)}_{=0} + 5B\underbrace{\cos(5\times 0)}_{=1} = 5 \quad \text{gives} \quad 5B = 5 \quad \Rightarrow \quad B = 1$$

Putting $A = 2$ and $B = 1$ into (☼) gives

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$$s = 2\cos(5t) + \sin(5t)$$

(b) We need to find s at $t = \frac{\pi}{4}$:

$$s = 2\cos\left(5\frac{\pi}{4}\right) + \sin\left(5\frac{\pi}{4}\right)$$

$$= 2\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}[2+1] = -\frac{3}{\sqrt{2}}$$

(c) The spring is initially at rest means that the acceleration $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$. Using $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + 25s = 0$ gives that $s = 0$.

4. The characteristic equation of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0.001x^2$ is

$$m^2 + 1 = 0 \quad \text{implies that} \quad m = \pm\sqrt{-1} = \pm j$$

The complementary function is given by

$$y_c = A\cos(x) + B\sin(x)$$

Since $f(x) = 0.001x^2$ we trail the particular integral

$$Y = Cx^2 + Dx + E$$

Differentiating this gives

$$\frac{dY}{dx} = 2Cx + D$$
$$\frac{d^2Y}{dx^2} = 2C$$

By substituting these into the given differential equation $\frac{d^2y}{dx^2} + y = 0.001x^2$ we can find the values of C, D and E. We have

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$$2C + Cx^2 + Dx + E = 0.001x^2$$
$$Cx^2 + Dx + (2C + E) = 0.001x^2 \qquad (\dagger)$$

Equating the x^2 coefficients of $(\dagger)$:

$$C = 0.001$$

Equating the x coefficients of $(\dagger)$:

$$D = 0$$

Equating the constant coefficients of $(\dagger)$:

$$2C + E = 0$$
$$2(0.001) + E = 0 \qquad [\text{Because } C = 0.001]$$
$$E = -0.002$$

Thus the particular integral is found by substituting these values $C = 0.001$, $D = 0$ and $E = -0.002$ into $Y = Cx^2 + Dx + E$ which gives

$$Y = 0.001x^2 - 0.002$$

Hence our general solution is given by

$$y = y_c + Y = A\cos(x) + B\sin(x) + 0.001x^2 - 0.002 \qquad (*)$$

We can find the values of A and B by using the given initial conditions

$$y(0) = 0, \quad y'(0) = 1.5$$

$y(0) = 0$ means that when $x = 0$, $y = 0$. Substituting this into (*) gives

$$A\underbrace{\cos(0)}_{=1} + B\underbrace{\sin(0)}_{=0} + 0.001(0)^2 - 0.002 = A - 0.002 = 0$$

Hence $A = 0.002$. To use the other initial condition $y'(0) = 1.5$ which means that when $x = 0$, $y' = 1.5$ so we need to differentiate (*):

$$y = A\cos(x) + B\sin(x) + 0.001x^2 - 0.002$$
$$y' = -A\sin(x) + B\cos(x) + 0.002x$$

Substituting $x = 0$ and $y' = 1.5$ into this yields

$$-A\underbrace{\sin(0)}_{=0} + B\underbrace{\cos(0)}_{=1} + 0.002(0) = B = 1.5$$

Our particular solution is found by substituting $A = 0.002$ and $B = 1.5$ into (*):

$$y = 0.002\cos(x) + 1.5\sin(x) + 0.001x^2 - 0.002$$
$$= 1.5\sin(x) + 0.001\left[2\cos(x) + x^2 - 2\right]$$

5. The characteristic equation of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = -4x$ is

$$m^2 + m - 2 = 0$$

Solving this quadratic equation:

$$m^2 + m - 2 = (m-1)(m+2) = 0 \quad \text{gives } m_1 = 1,\ m_2 = -2$$

Our complementary function is $y_c = Ae^x + Be^{-2x}$. *What is our trail function in this case?*

Since we have $f(x) = -4x$ therefore $Y = Cx + D$. Differentiating this:

$$Y = Cx + D$$
$$Y' = C$$
$$Y'' = 0$$

Substituting these results into the given differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4x$ yields

$$0 + C - 2(Cx + D) = -4x$$
$$C - 2Cx - 2D = -4x$$

Equating coefficients of *x*:

$$-2C = -4 \text{ implies that } C = 2$$

Equating constants:

$$C - 2D = 0$$
$$2 - 2D = 0 \quad \text{implies that } D = \frac{2}{2} = 1$$

Our particular integral is $Y = Cx + D = 2x + 3$. This means that our general solution is

$$y = y_c + Y = Ae^x + Be^{-2x} + 2x + 1 \qquad (*)$$

We need to find the particular solution which satisfies the given initial conditions $y(0) = 4$, $y'(0) = 5$. What does $y(0) = 4$ mean?

When $x = 0$, $y = 4$. Substituting these into (*) yields

$$Ae^0 + Be^{-(2\times 0)} + (2\times 0) + 1 = A + B + 0 + 1 = 4$$
$$A + B = 4 - 1 = 3$$

To use the other initial condition we need to differentiate (*):

$$y' = Ae^x - 2Be^{-2x} + 2$$

Applying the initial condition $y'(0) = 5$ we have

$$Ae^0 - 2Be^{-(2\times 0)} + 2 = A - 2B + 2 = 5$$
$$A - 2B = 5 - 2 = 3$$

Solving the simultaneous equations

$$\left.\begin{aligned} A + B &= 3 \\ A - 2B &= 3 \end{aligned}\right\} \quad \text{implies that } A = 3,\ B = 0$$

Our particular solution is given by substituting these values of $A = 3$ and $B = 0$ into (*):

$$y = Ae^x + Be^{-2x} + 2x + 1$$
$$= 3e^x - \left(0 \times e^{-2x}\right) + 2x + 1 = 3e^x + 2x + 1$$

6. (a) The characteristic equation of $2y'' + 5y' + 3y = 0$ is

$$2m^2+5m+3=0$$

$$(2m+3)(m+1)=0 \quad \text{implies that } m_1=-\frac{3}{2},\ m_2=-1$$

Our general solution is

$$y=Ae^{-\frac{3}{2}x}+Be^{-x} \qquad (*)$$

Substituting the given initial condition $y(0)=3$ [when $x=0,\ y=3$] into (*):

$$Ae^{-\frac{3}{2}\times 0}+Be^{0}=A+B=3$$

We need to differentiate (*) to use the other initial condition:

$$y=Ae^{-\frac{3}{2}x}+Be^{-x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{2}Ae^{-\frac{3}{2}x}-Be^{-x} \qquad \left[\text{Using } \left(e^{kx}\right)'=ke^{kx}\right]$$

Applying the other initial condition $y'(0)=-4$ which means that when $x=0$, $y'=-4$:

$$-\frac{3}{2}Ae^{-\frac{3}{2}\times 0}-Be^{0}=-\frac{3}{2}A-B=-4$$

We need to solve the simultaneous equations

$$\left.\begin{aligned}A+B&=3\\ -\frac{3}{2}A-B&=-4\end{aligned}\right\}\ \text{gives } A=2 \text{ and } B=1$$

The particular solution is determined by substituting $A=2$ and $B=1$ into (*):

$$y=Ae^{-\frac{3}{2}x}+Be^{-x}=2e^{-\frac{3}{2}x}+e^{-x}$$

The solution is $y=2e^{-\frac{3}{2}x}+e^{-x}$.

(b) First we find the characteristic equation of $y''-y'=\sin(2x)$:

$$m^2-m=0$$

$$m(m-1)=0 \quad m_1=0,\ m_2=1$$

The complementary function is given by

$$y_c=Ae^{0}+Be^{x}=A+Be^{x} \qquad \left[\text{Because } e^{0}=1\right]$$

Our trail function is $Y=C\cos(2x)+D\sin(2x)$. We need to differentiate this in order to find the values of *C* and *D*.

$$Y=C\cos(2x)+D\sin(2x)$$

$$\frac{\mathrm{d}Y}{\mathrm{d}x}=-2C\sin(2x)+2D\cos(2x)$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}=-4C\cos(2x)-4D\sin(2x)$$

Putting this into the given differential equation $y''-y'=\sin(2x)$:

$$-4C\cos(2x)-4D\sin(2x)-\left[-2C\sin(2x)+2D\cos(2x)\right]=\sin(2x)$$

$$(-4C-2D)\cos(2x)+(2C-4D)\sin(2x)=\sin(2x)$$

Equating coefficients of $\cos(2x)$:

$$-4C-2D=0$$

Equating coefficients of $\sin(2x)$:

$$2C-4D=1$$

Solving these simultaneous equations:

$$\left.\begin{array}{r}-4C-2D=0\\2C-4D=1\end{array}\right\}\Rightarrow\quad C=\frac{1}{10},\ D=-\frac{1}{5}$$

The particular integral is

$$Y=C\cos(2x)+D\sin(2x)$$

$$=\frac{1}{10}\cos(2x)-\frac{1}{5}\sin(2x)=\frac{1}{10}\left[\cos(2x)-2\sin(2x)\right]$$

The general solution is given by

$$y=y_c+Y=A+Be^x+\frac{1}{10}\left[\cos(2x)-2\sin(2x)\right]$$

7. (a) We can test the function $y=\frac{1}{2}x\sin(x)$ is a solution of the given differential equation by differentiating this and then substituting the results into the differential equation.

$$y=\frac{1}{2}x\sin(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\underbrace{\left[\sin(x)+x\cos(x)\right]}_{\text{Using the product rule}}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}\left[\cos(x)+\cos(x)-x\sin(x)\right]=\frac{1}{2}\left[2\cos(x)-x\sin(x)\right]$$

Substituting these results into the LHS of $y''+y=\sin(x)$ gives

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$$y''+y=\frac{1}{2}\left[2\cos(x)-x\sin(x)\right]+\frac{1}{2}x\sin(x)$$

$$=\cos(x)$$

Thus $y=\frac{1}{2}x\sin(x)$ is **not** a solution of the given differential equation $y''+y=\sin(x)$.

(b) This time we test $y=-\frac{1}{2}x\cos(x)$. We have

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$$y=-\frac{1}{2}x\cos(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}\left[\cos(x)-x\sin(x)\right]\qquad\qquad[\text{By product rule}]$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{2}\left[-\sin(x)-\left(\sin(x)+x\cos(x)\right)\right]$$

$$=-\frac{1}{2}\left[-2\sin(x)-x\cos(x)\right]=\frac{1}{2}\left[2\sin(x)+x\cos(x)\right]$$

Substituting this into the LHS of $y''+y=\sin(x)$ gives

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$$y'' + y = \frac{1}{2}\left[2\sin(x) + x\cos(x)\right] - \frac{1}{2}x\cos(x)$$
$$= \sin(x)$$

This $y = -\frac{1}{2}x\cos(x)$ is a solution of the given differential equation.

8. We need to solve $\frac{d^2 y}{dx^2}+\frac{dy}{dx}-2y=e^{3x}$. We first find the complementary function:

$$m^2+m-2=0$$

$$(m-1)(m+2)=0 \quad \text{gives } m_1=1,\ m_2=-2$$

Our complementary function is equal to

$$y_c=Ae^{x}+Be^{-2x}$$

Because $f(x)=e^{3x}$ therefore our trail function is $Y=Ce^{3x}$. Differentiating this gives

$$\frac{dY}{dx}=3Ce^{3x} \qquad \left[\text{Because } \left(e^{kx}\right)'=ke^{kx}\right]$$

$$\frac{d^2Y}{dx^2}=9Ce^{3x} \qquad \left[\text{Because } \left(e^{kx}\right)'=ke^{kx}\right]$$

Substituting these into the given differential equation $\frac{d^2 y}{dx^2}+\frac{dy}{dx}-2y=e^{3x}$:

$$9Ce^{3x}+3Ce^{3x}-2Ce^{3x}=e^{3x}$$

$$10Ce^{3x}=e^{3x} \quad \text{implies that } C=\frac{1}{10}$$

The particular integral is $Y=Ce^{3x}=\frac{1}{10}e^{3x}$. The general solution is given by

$$y=y_c+Y=Ae^{x}+Be^{-2x}+\frac{1}{10}e^{3x}$$

The solution to the given differential equation is $Ae^{x}+Be^{-2x}+\frac{1}{10}e^{3x}$.

9. The characteristic equation is

$$m^2+1=0 \quad \text{implies that } m=\pm\sqrt{-1}=\pm j$$

Our complementary function y_c is Deleted:

$$y_c=A\cos(x)+B\sin(x)$$

The trial function in this case is

$$Y=x\left[C\cos(x)+D\sin(x)\right]$$

Differentiating this function by using the product rule we have

$$\frac{dY}{dx}=1\left[C\cos(x)+D\sin(x)\right]+x\left[-C\sin(x)+D\cos(x)\right]$$

$$=\left[C+Dx\right]\cos(x)+\left[D-Cx\right]\sin(x)$$

$$\frac{d^2Y}{dx^2}=D\cos(x)+\left[C+Dx\right]\left[-\sin(x)\right]+(-C)\sin(x)+\left[D-Cx\right]\cos(x)$$

$$=\left[2D-Cx\right]\cos(x)-\left[2C+Dx\right]\sin(x)$$

Substituting this into the given differential equation $y''+y=\cos x$:

$$\left[2D-Cx\right]\cos(x)-\left[2C+Dx\right]\sin(x)+x\left[C\cos(x)+D\sin(x)\right]=\cos(x)$$

$$2D\cos(x)-2C\sin(x)=\cos(x)$$

From the last equation we have $D=\frac{1}{2}$ and $C=0$. This means our particular integral is

$$\begin{aligned} Y &= x\left[C\cos(x)+D\sin(x)\right] \\ &= x\left[(0)\cos(x)+\frac{1}{2}\sin(x)\right]=\frac{1}{2}x\sin(x) \end{aligned}$$

Thus our general solution is

$$y=y_c+Y=A\cos(x)+B\sin(x)+\frac{1}{2}x\sin(x) \qquad (*)$$

We can find a particular solution because we have been given initial conditions. Substituting the initial condition $y(0)=0$ which means that when $x=0$ then $y=0$:

$$A\underbrace{\cos(0)}_{=1}+B\underbrace{\sin(0)}_{=0}+\frac{1}{2}(0)\sin(0)=A=0$$

The other initial condition is $y'(0)=5/2$ means that when $x=0$, $y'=5/2$. We need to differentiate (*) in order to use this condition:

$$y'=-A\sin(x)+B\cos(x)+\frac{1}{2}\underbrace{\left[\sin(x)+x\cos(x)\right]}_{\text{Using the product rule}}$$

Substituting $x=0$ and $y'=5/2$ into this yields:

$$-A\underbrace{\sin(0)}_{=0}+B\underbrace{\cos(0)}_{=1}+\frac{1}{2}\left[\sin(0)+0\cos(0)\right]=B=\frac{5}{2}$$

Hence our particular solution is given by putting $A=0$ and $B=\frac{5}{2}$ into (*):

$$\begin{aligned} y &= A\cos(x)+B\sin(x)+\frac{1}{2}x\sin(x) \\ &= 0\cos(x)+\frac{5}{2}\sin(x)+\frac{1}{2}x\sin(x) \\ &= \frac{1}{2}\left[5\sin(x)+x\sin(x)\right] \end{aligned}$$

Our solution is $y=\frac{1}{2}\left[5\sin(x)+x\sin(x)\right]$.

10. The characteristic equation is given by

$$m^2+9=0 \quad \text{gives} \quad m=\pm j3$$

Our complementary function is

$$y_c=A\cos(3t)+B\sin(3t)$$

Our trial function is

$$Y=Ct^2+Dt+E+Ft\cos(3t)+Gt\sin(3t)$$

We need to find the values of the unknowns C, D, E, F and G. *How?*

By differentiating twice and substituting into the given differential equation:

$$\frac{dY}{dx} = 2Ct + D + F\left[(1)\cos(3t) - 3t\sin(3t)\right] + G\left[(1)\sin(3t) + 3t\cos(3t)\right]$$
$$= 2Ct + D + F\cos(3t) - 3tF\sin(3t) + G\sin(3t) + 3Gt\cos(3t)$$
$$\frac{d^2Y}{dx^2} = 2C - 3F\sin(3t) - 3F\left[(1)\sin(3t) + 3t\cos(3t)\right] + 3G\cos(3t) + 3G\left[(1)\cos(3t) - 3t\sin(3t)\right]$$
$$= 2C - 3F\sin(3t) - 3F\sin(3t) - 9tF\cos(3t) + 3G\cos(3t) + 3G\cos(3t) - 9Gt\sin(3t)$$
$$= 2C - 6F\sin(3t) - 9tF\cos(3t) + 6G\cos(3t) - 9Gt\sin(3t)$$

Substituting these results into the differential equation yields $\frac{d^2y}{dt^2} + 9y = 9t^2 - 12\cos(3t)$

$$2C - 6F\sin(3t) - 9tF\cos(3t) + 6G\cos(3t) - 9Gt\sin(3t)$$
$$+ 9\left(Ct^2 + Dt + E + Ft\cos(3t) + Gt\sin(3t)\right) = 9t^2 - 12\cos(3t)$$
$$2C - 6F\sin(3t) - 9tF\cos(3t) + 6G\cos(3t) - 9Gt\sin(3t)$$
$$+ 9Ct^2 + 9Dt + 9E + 9Ft\cos(3t) + 9Gt\sin(3t) = 9t^2 - 12\cos(3t)$$
$$2C - 6F\sin(3t) + 6G\cos(3t) + 9Ct^2 + 9Dt + 9E = 9t^2 - 12\cos(3t) \qquad (\dagger)$$

Equating coefficients of t^2 in (†):

$$9C = 9 \quad \text{gives} \quad C = 1$$

Equating coefficients of t in (†):

$$9D = 0 \quad \text{gives} \quad D = 0$$

Equating coefficients of constants in (†):

$$2C + 9E = 0$$

From above we have $C = 1$ therefore $2C + 9E = 2 + 9E = 0$ implies that $E = -\frac{2}{9}$.

Equating coefficients of $\cos(3t)$ in (†):

$$6G = -12 \quad \text{gives} \quad G = -2$$

Equating coefficients of $\sin(3t)$ in (†):

$$-6F = 0 \quad \text{gives} \quad F = 0$$

We have $C = 1$, $D = 0$, $E = -\frac{2}{9}$, $F = 0$ and $G = -2$. Thus our particular integral is

$$Y = Ct^2 + Dt + E + Ft\cos(3t) + Gt\sin(3t)$$
$$= t^2 + (0)t - \frac{2}{9} + (0)t\cos(3t) - 2t\sin(3t)$$
$$= t^2 - \frac{2}{9} - 2t\sin(3t)$$

Our general solution is

$$y = y_c + Y = A\cos(3t) + B\sin(3t) + t^2 - \frac{2}{9} - 2t\sin(3t)$$

11. (a) We need to solve $\ddot{x} = -9x$ which has the characteristic equation given by

$$m^2 = -9 \quad \text{implies that} \quad m = \pm\sqrt{-9} = \pm j3$$

Our general solution is

$$x = A\cos(3t) + B\sin(3t) \qquad (☼)$$

Substituting the given initial condition $x = -1$ when $t = 0$ into (☼):

$$A\underbrace{\cos(3\times 0)}_{=1} + B\underbrace{\sin(3\times 0)}_{=0} = A = -1$$

To use the other initial condition $\dot{x} = 3$ when $t = 0$ we need to differentiate (☼):

$$x = A\cos(3t) + B\sin(3t)$$

$$\dot{x} = -3A\sin(3t) + 3B\cos(3t)$$

Substituting $t = 0$, $\dot{x} = 3$ into this:

$$-3A\underbrace{\sin(3\times 0)}_{=0} + 3B\underbrace{\cos(3\times 0)}_{=1} = 3B = 3 \quad \text{gives} \quad B = 1$$

Our particular solution is found by putting $A = -1$ and $B = 1$ into (☼):

$$x = A\cos(3t) + B\sin(3t) = -\cos(3t) + \sin(3t)$$

We need to place this into amplitude-phase form $R\sin(\omega t + \phi)$. In general

$$a\cos(\theta) + b\sin(\theta) = R\sin(\theta + \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \alpha = \tan^{-1}\left(\frac{a}{b}\right)$$

For $x = -\cos(3t) + \sin(3t)$ we have $a = -1$ and $b = 1$ therefore $R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $\alpha = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$. We have

$$x = -\cos(3t) + \sin(3t) = \sqrt{2}\sin\left(3t - \frac{\pi}{4}\right)$$

The amplitude is $R = \sqrt{2}$ and period $\frac{2\pi}{3}$.

To sketch the graph of $x = \sqrt{2}\sin\left(3t - \frac{\pi}{4}\right) = \sqrt{2}\sin\left[3\left(t - \frac{\pi}{12}\right)\right]$ is the sine graph with amplitude of $\sqrt{2}$ and covering 3 cycles between 0 to 2π and shifted to the right by $\frac{\pi}{12}$ rad:

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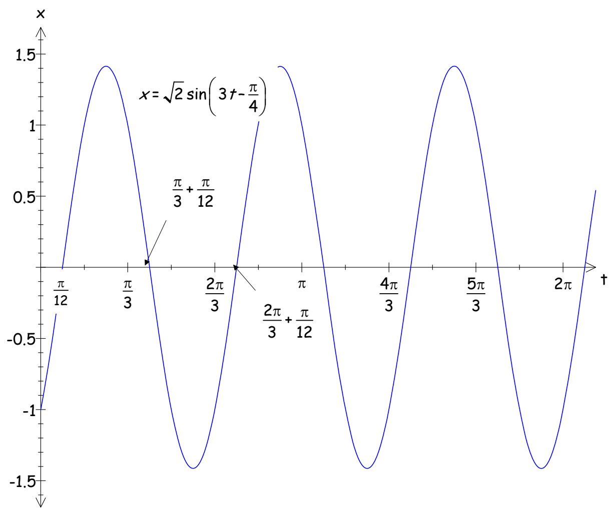

(b) We first find the homogeneous solution of $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 10x = 20t + 6$ which means we have zero on the RHS:

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$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 10x = 0$$

The characteristic equation is given by

$$m^2 - 2m + 10 = 0$$

To solve this quadratic equation we need to use the formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 10)}}{2}$$

$$= \frac{2 \pm \sqrt{-36}}{2}$$

$$= \frac{2}{2} \pm j\frac{6}{2} = 1 \pm j3$$

Our complementary function is given by

$$x_c = e^t\left[A\cos(3t) + B\sin(3t)\right]$$

Our trial function is $X = Ct + D$. Differentiating this gives

$$X = Ct + D$$

$$X' = C$$

$$X'' = 0$$

Substituting these into the given differential equation $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 10x = 20t + 6$:

$$0 - 2C + 10(Ct + D) = 10Ct - 2C + 10D = 20t + 6$$

Equating coefficients of t:

$$10C = 20 \text{ gives } C = 2$$

Equating constants:

$$-2C + 10D = -2(2) + 10D = 6 \quad \text{implies that } D = 1$$

The particular integral is equal to $X = 2t + 1$. Hence our solution is given by

$$x = x_c + X$$

$$= e^t\left[A\cos(3t) + B\sin(3t)\right] + 2t + 1$$