Complete Solutions to Examination Questions 16

1. The simplest way to evaluate the standard deviation and mean is to use these functions on your calculator. However below we take the long route by working with the definitions. The mean is given by adding all data and dividing by the number of data:

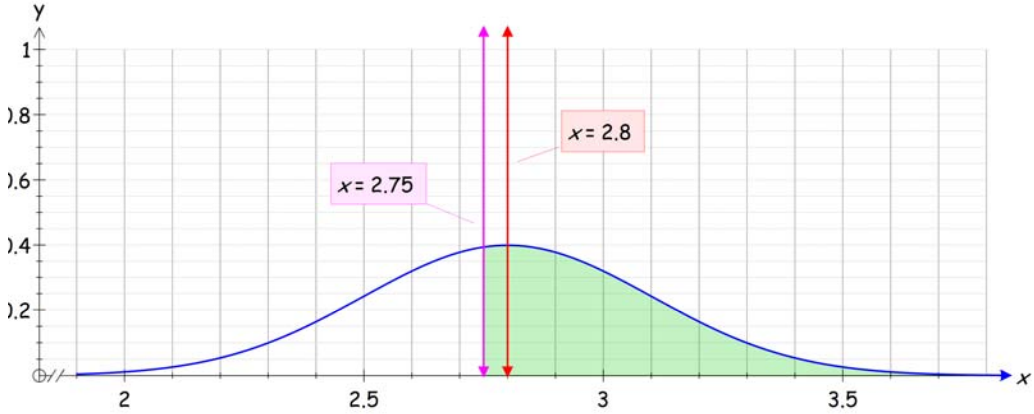
$$\text{Mean} = \frac{2.35+3.21+1.85+4.01+2.73+3.11+2.16+3.14}{8} = 2.82$$

The variance is given by

$$\text{Variance} = \frac{(2.35-2.82)^2+(3.21-2.82)^2+(1.85-2.82)^2+\ldots+(3.14-2.82)^2}{8} = 0.42$$

The standard deviation is the square root of this, $s.d. = \sqrt{0.42} = 0.648$.

2. We need to use the normal distribution table at the back of the book. Let *x* denote lengths then we have the following representation of lengths that are longer than 2.75m:



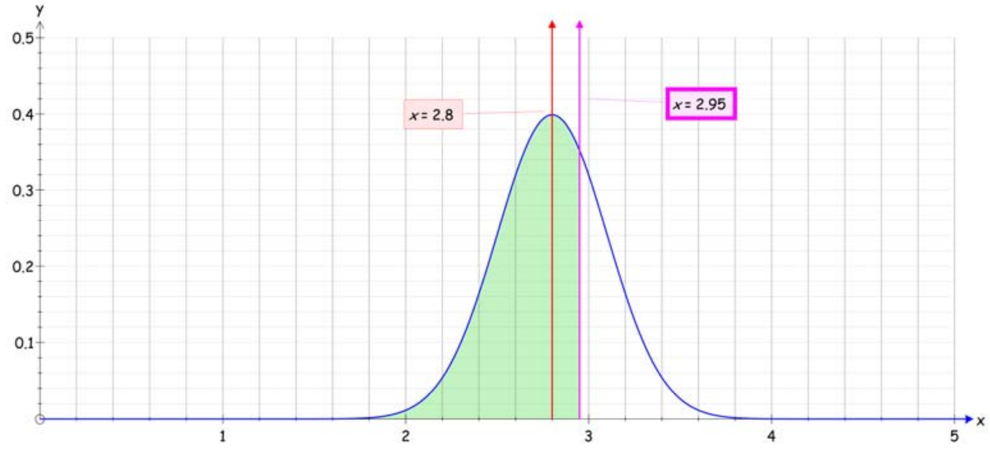
The normal distribution variable $z = \frac{x-\mu}{\sigma}$ where $\mu =$ mean and $\sigma =$ s.d. In our case we have $\mu = 2.8$ and $\sigma = 0.13$. Hence $z = \frac{x-2.8}{0.13}$.

For rods longer than 2.75m we have:

$$z = \frac{2.75-2.8}{0.13} = -0.3846$$

By using the normal distribution table we need to find $P(z > -0.3846)$. Remember the minus sign only signifies that the *z* is less than $z = 0$ which means 2.75 is less than the mean of 2.8. Normal distribution table gives $P(z > -0.3846) = 0.6499$. This means that the proportion of rods longer than 2.75m is 0.65 (2dp).

Similarly for lengths less than 2.95m we have the following representation of data:



For rod lengths less than 2.95m we have:

$$z=\frac{2.95-2.8}{0.13}=1.1538$$

We need to find $P\left(z<1.1538\right)$. Using the table gives

$$P\left(z<1.1538\right)=0.8757$$

The proportion of rods less than 2.95m is 0.88 (2dp).

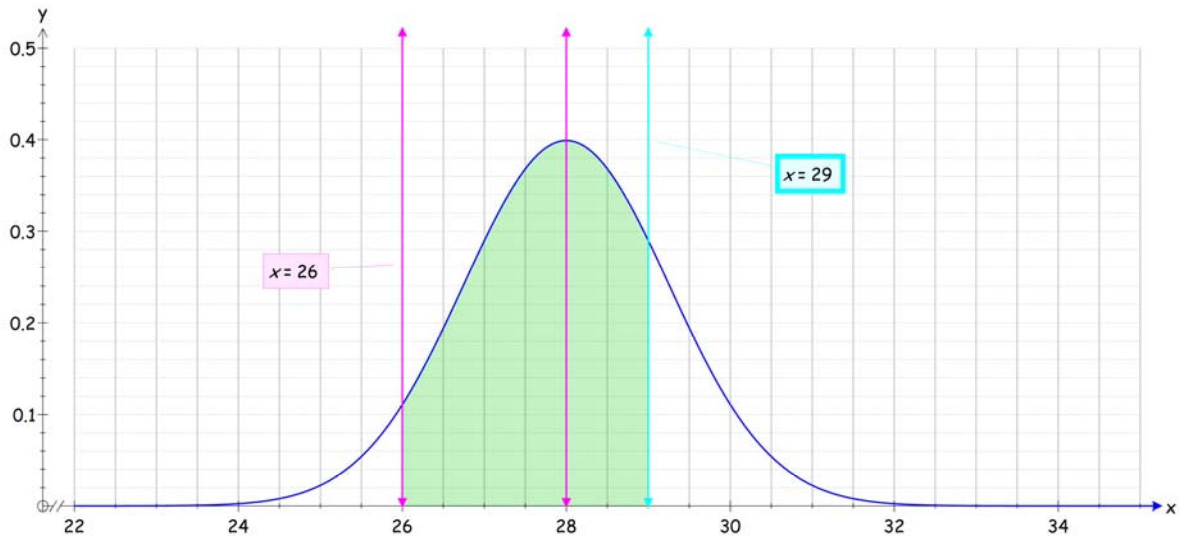
3. i) We have the following table:

Speed (mph)	Midpoint	Frequency
20-23	21.5	5
23-26	24.5	4
26-29	27.5	8
29-32	30.5	22
32-35	33.5	6
35-45	40	5

ii) and iii) We can use a calculator to find the mean and standard deviation. You can access the instructions from the handbook of the calculator or the web. We have

$$\text{Mean}=29.95,\ \ sd=4.714$$

iv) The proportion of cars which are driving between 26mph and 29mph can be represented by the shaded area below (*x* is the speed):



The z variable of the normal distribution is given by

$$z = \frac{x-\mu}{\sigma} = \frac{x-28}{1.25} \qquad \left[\text{Because we are given mean is 28 and sd is 1.25}\right]$$

For $x = 26$ we have

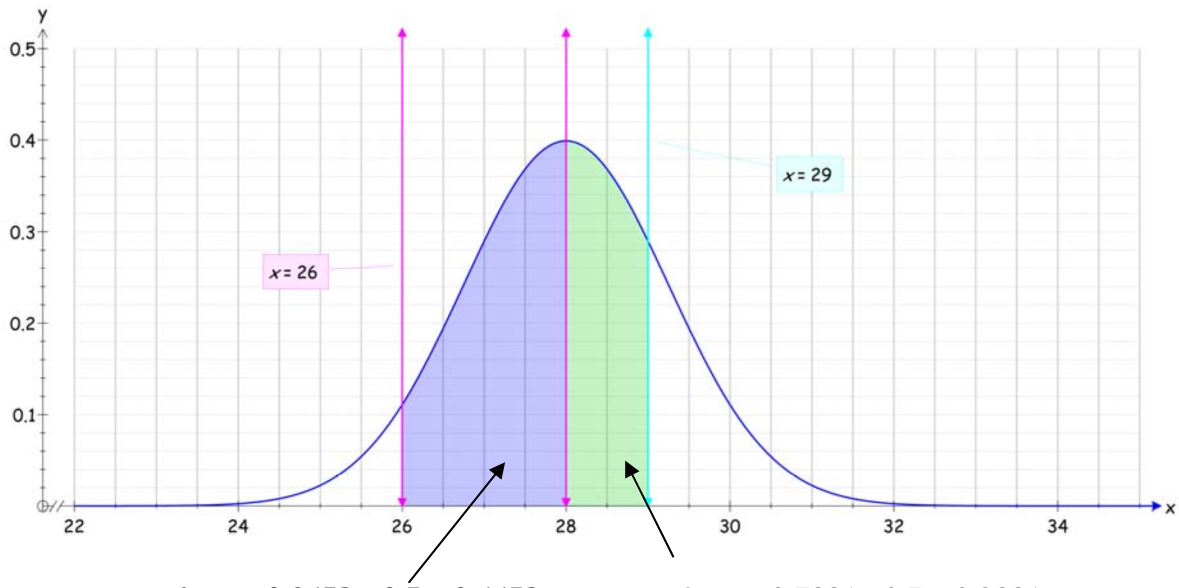
$$z_1 = \frac{26-28}{1.25} = -1.6$$

The normal distribution table gives an area of 0.9452 for $P(z > 1.6)$.

For $x = 29$ we have

$$z_2 = \frac{29-28}{1.25} = 0.8$$

The normal distribution table gives $P(z < 0.8) = 0.7881$.



$Area = 0.9452 - 0.5 = 0.4452$ $\qquad$ $Area = 0.7881 - 0.5 = 0.2881$

Hence the proportion of cars between 26mph and 29mph is $0.4452 + 0.2881 = 0.7333$ which means the number of cars driving between these speeds is

$$50 \times 0.7333 = 36.665$$

You expect about 37 out of the 50 cars to be driving between the speeds of 26mph and 29mph.

v) In the given sample most (22 out of 50) of the cars drive between 29mph and 32mph and only 8 out 50 cars drive between 26mph and 29mph.

4. The probability of selecting a blue ball is $P(Blue) = \frac{5}{12}$, the probability of selecting a red ball is $P(red) = \frac{4}{12}$ and the probability of selecting a yellow ball is $P(yellow) = \frac{3}{12}$.

$$P(\text{2 blue balls}) = \frac{5}{12} \times \frac{4}{11} = \frac{5}{33}$$

$$P(\text{2 red balls}) = \frac{4}{12} \times \frac{3}{11} = \frac{1}{11}$$

$$P(\text{2 yellow balls}) = \frac{3}{12} \times \frac{2}{11} = \frac{1}{22}$$

The probability of selecting two balls of the same colour

$$P(2\text{ same colour}) = \frac{5}{33} + \frac{1}{11} + \frac{1}{22} = \frac{19}{66}$$

5. The probability of a Poisson distribution is given by $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}$. We are given that average is 4 therefore $\mu = 4$ and our distribution has the formula:

$$P(X = x) = \frac{e^{-4}4^x}{x!}$$

The probability of more than three faults is $1 - P(X \le 3)$. We need to find $P(X = 0)$, $P(X = 1)$, $P(X = 2)$ and $P(X = 3)$:

$$P(X = 0) = \frac{e^{-4}4^0}{0!} = 0.0183$$

$$P(X = 1) = \frac{e^{-4}4^1}{1!} = 0.0733$$

$$P(X = 2) = \frac{e^{-4}4^2}{2!} = 0.1465$$

$$P(X = 3) = \frac{e^{-4}4^3}{3!} = 0.1954$$

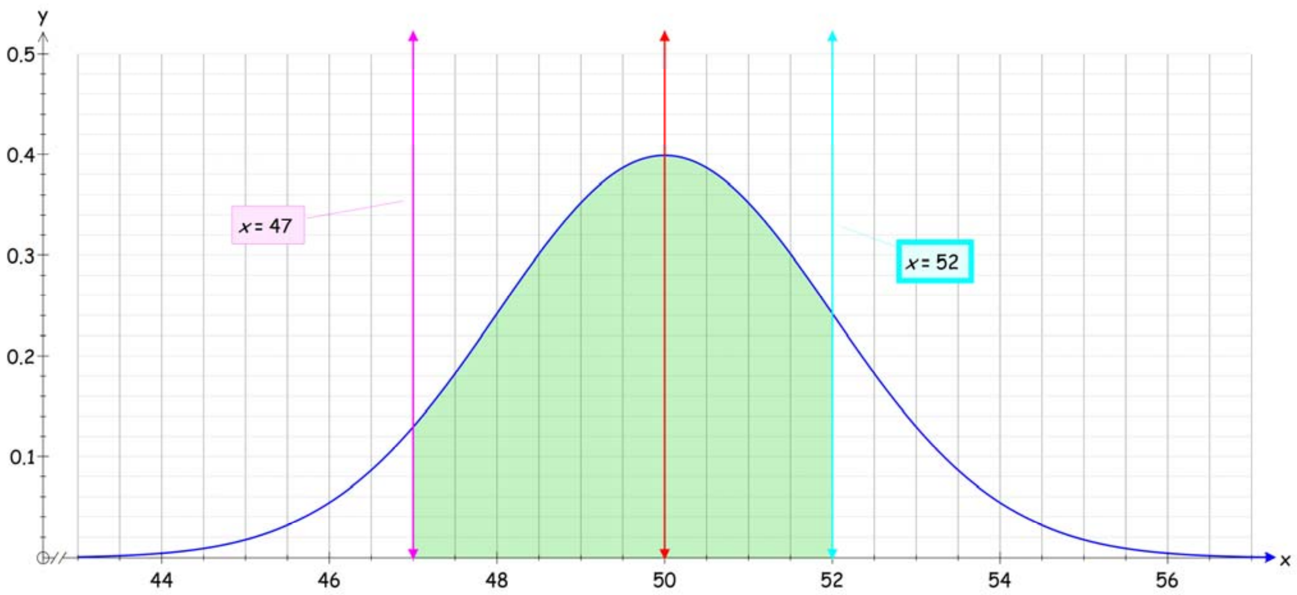
The probability of more than three faults is

$$P(X > 3) = 1 - P(X \le 3)$$

$$= 1 - [0.0183 + 0.0733 + 0.1465 + 0.1954] = 0.5665$$

6. We are given that $\mu = 50$ and $\sigma = 2$.

(i) The distribution of the weights x is a normal distribution. The shaded area below represents the bars which weigh between 47g and 52g:



We need to find the shaded area by using the normal distribution table. The z variable is given by $z=\frac{x-\mu}{\sigma}=\frac{x-50}{2}$. For $x=47$ we have

$$z_1=\frac{47-50}{2}=-1.5$$

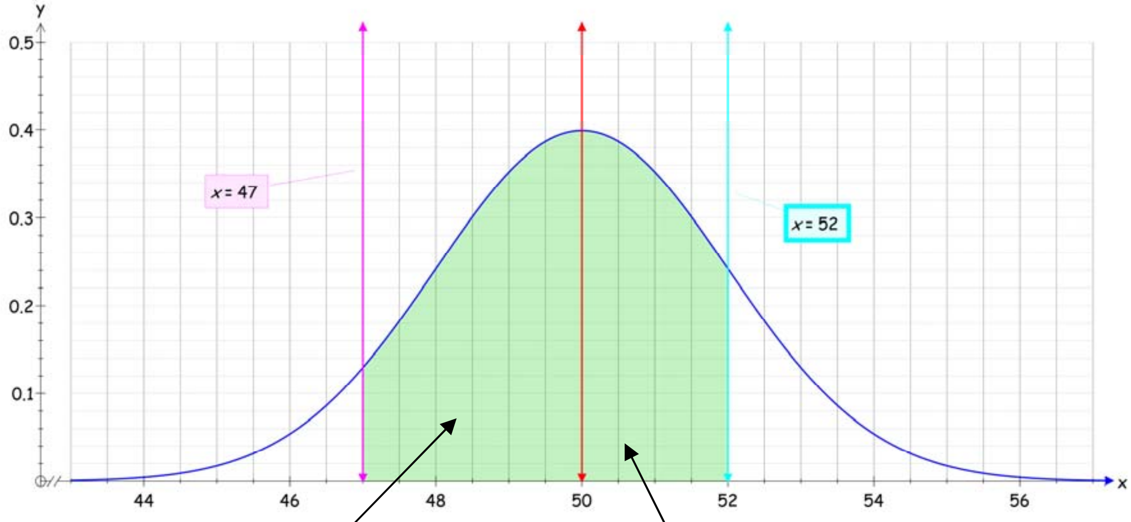
Looking up the normal distribution table for $z_1=-1.5$:

$$P(z>-1.5)=0.9332$$

For $x=52$ we have $z_2=\frac{52-50}{2}=1$. Similarly

$$P(z<1)=0.8413$$

Subtracting 0.5 from each of these values gives



$0.8413-0.5=0.3413$

$0.9332-0.5=0.4332$

The probability that a bar weighs between 47g and 52g is $0.4332+0.3413=0.7745$.
(ii) Probability that at least 3 will be rejected is

$$P(\text{at least 3 will be rejected})=1-P[\text{less than 3 will be rejected}]$$

Note that $P(\text{less than 3 will be rejected})=P(X=0)+P(X=1)+P(X=2)$. Therefore

$$P(\text{at least 3 will be rejected})=1-[P(X=0)+P(X=1)+P(X=2)] \qquad (*)$$

By using the above result of part (i) the probability that a bar will be rejected is $1-0.7745=0.2255$.
Applying the binomial distribution formula we have

$$P(X=0)=0.7745^{10}=0.07766$$

$$P(X=1)=C_1^{10}\times 0.7745^9\times 0.2255=10\times 0.7745^9\times 0.2255=0.22612$$

$$P(X=2)=C_2^{10}\times 0.7745^8\times 0.2255^2=45\times 0.7745^8\times 0.2255^2=0.29626$$

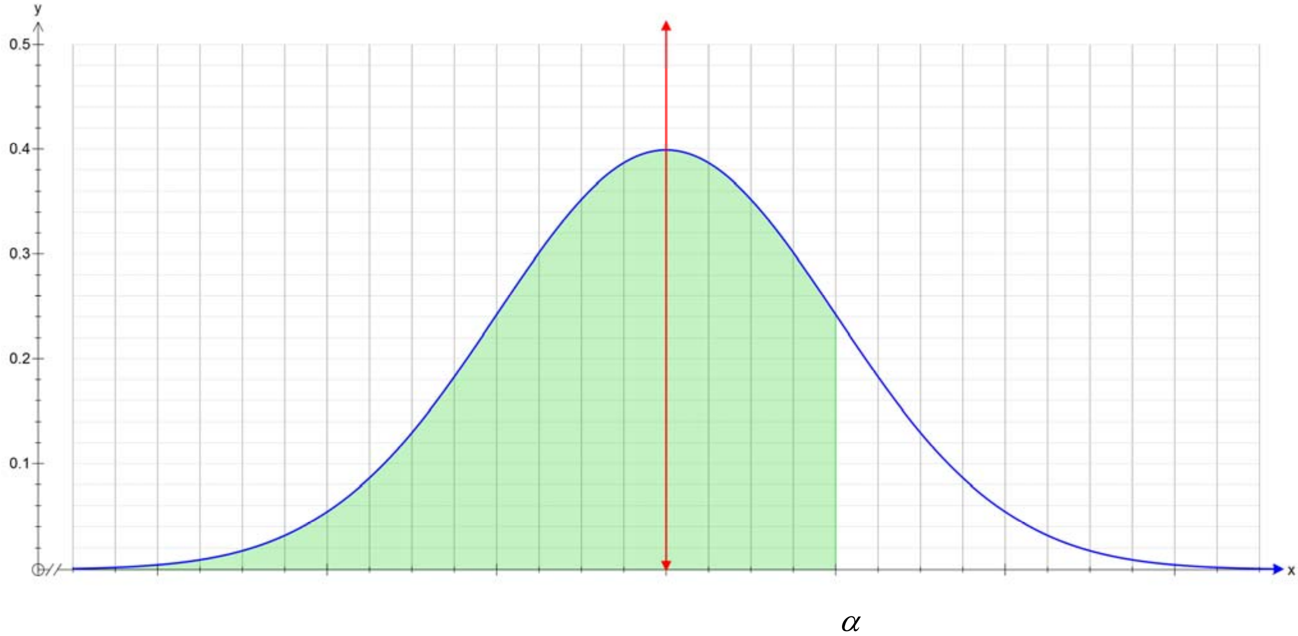
Substituting these into (*) gives

$$P(\text{at least 3 will be rejected})=1-[P(X=0)+P(X=1)+P(X=2)]$$

$$=1-[0.07766+0.22612+0.29626]=0.39996$$

The probability that less than three will be rejected is 0.4 (1dp).

(iii) We need to find α such that the area of the normal distribution less than α grams is 0.8. This can be represented graphically as:



We need to find the z value for 0.8 by using the normal distribution table. Thus $z = 0.842$ which means that

$$\frac{\alpha - 50}{2} = 0.842 \text{ re-arranging gives } \alpha = (2 \times 0.842) + 50 = 51.684$$

The probability that a bar weighs less than $\alpha = 51.684\,\text{g}$ is 0.8.

7. (a) The mean $\mu = np = 50 \times 0.03 = 1.5$. The Poisson distribution formula is given by

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!} = \frac{e^{-1.5}1.5^x}{x!}$$

The probability that the box contains at least two defective bolts is

$$P(\text{at least 2 defective bolts}) = 1 - P(\text{less than 2 defective bolts}) \qquad (\dagger)$$

$P(\text{less than 2 defective bolts}) = P(X = 0) + P(X = 1)$. We have

$$\begin{aligned} P(\text{less than 2 defective bolts}) &= P(X = 0) + P(X = 1) \\ &= \frac{e^{-1.5}1.5^0}{0!} + \frac{e^{-1.5}1.5^1}{1!} = 0.2231 + 0.3347 = 0.5578 \end{aligned}$$

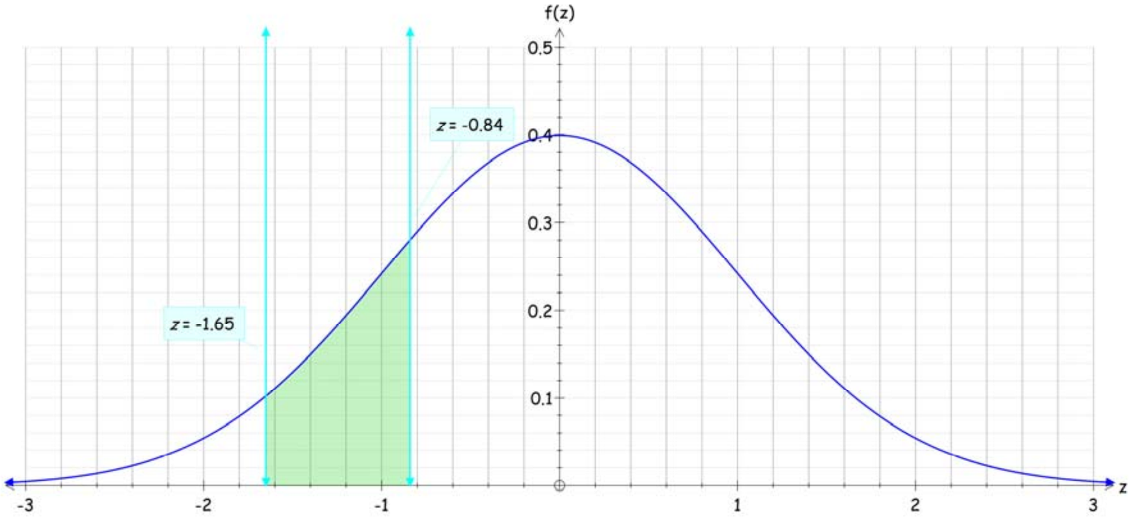
Substituting this $P(\text{less than 2 defective bolts}) = 0.5578$ into (†) gives

$$P(\text{at least 2 defective bolts}) = 1 - 0.5578 = 0.4422$$

(b) i. We need to use the normal distribution table with $z = 2$. We have

$$P(z < 2) = 0.9772$$

ii. Between $z = -1.65$ and $z = -0.84$. We can draw a normal distribution graph and the area which lies between $z = -1.65$ and $z = -0.84$ is shaded area below:



The normal distribution table gives

$$P(z > -1.65) = 0.9505 \text{ and } P(z > -0.84) = 0.7995$$

The shaded area is given by

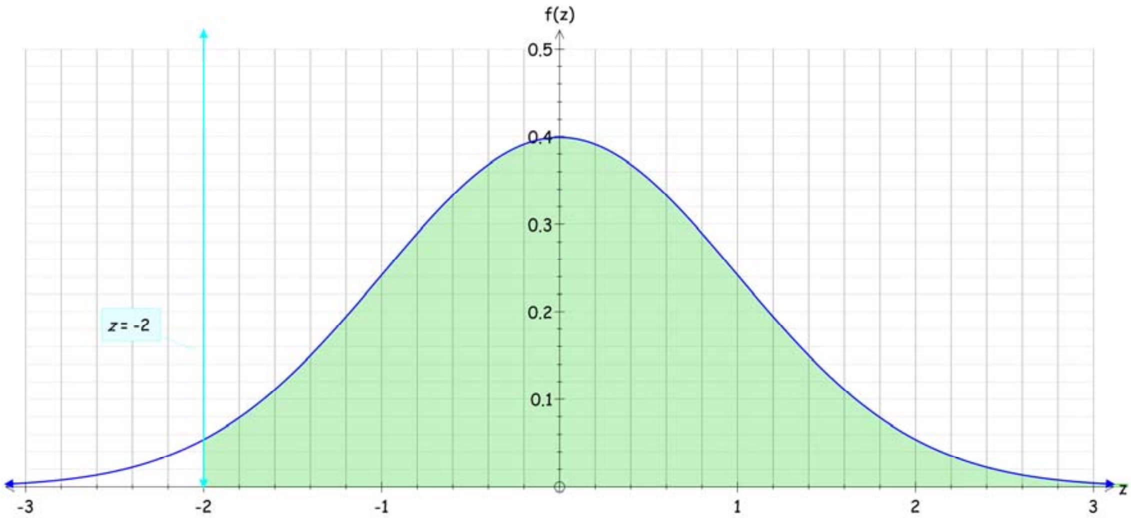
$$0.9505 - 0.7995 = 0.151$$

The probability that the random variable lies between $z = -1.65$ and $z = -0.84$ is 0.151.

(c) The z variable is given by $z = \dfrac{x - \mu}{\sigma}$. We are given that mean, $\mu = 1400$, and standard deviation, $\sigma = 200$, therefore $z = \dfrac{x - \mu}{\sigma} = \dfrac{x - 1400}{200}$. Substituting $x = 1000$ yields

$$z = \frac{1000 - 1400}{200} = -2$$

On a normal distribution graph we have:



The shaded area (or probability) can be found by using the normal distribution table,

$$P(z > -2) = 0.9772$$

The probability that a random bulb fails to meet the guarantee is $1 - 0.9772 = 0.0228$.

(d) We first determine the probability that a link exceeds 490 units. We are given that mean $\mu = 500$ and standard deviation $\sigma = 10$. The z variable is $z = \frac{x-\mu}{\sigma} = \frac{x-500}{10}$. Since we want to find the probability that a link exceeds 490 units therefore we substitute $x = 490$:

$$z = \frac{490-500}{10} = -1$$

Using the Normal distribution table gives $P(z > 1) = 0.8413$. Hence

$$P(\text{link exceeds 490 units}) = 0.8413$$

The probability that the strength of the chain of three links exceeds 490 units is

$$0.8413^3 = 0.5955$$

8. (a) (i) Probability that a valve is faulty is given by $P(\text{faulty}) = \frac{2}{20} = \frac{1}{10} = 0.1$. Let

The probability that the valve is working is $1 - 0.1 = 0.9$. Hence

$$P(\text{none are faulty}) = 0.9^4 = 0.6561$$

(ii) The probability that at least one is faulty is given by

$$P(\text{at least one is faulty}) = 1 - P(\text{none are faulty})$$

Substituting the result of part (i), $P(\text{none are faulty}) = 0.6561$, into this yields

$$P(\text{at least one is faulty}) = 1 - 0.6561 = 0.3439$$

(b) Probability that a randomly chosen component is faulty is $P(\text{faulty}) = \frac{3}{100} = 0.03$.

(i) $P(\text{working order}) = 0.97$. The probability that all 10 components are in working order is given by

$$P(\text{that all 10 components are in working order}) = 0.97^{10} = 0.7374$$

(ii) The probability that 9 components are in working order is

$$P(\text{9 components are in working order}) = 9 \times 0.97^9 \times 0.03 = 0.2052$$

(iii) The probability that at most one of the components is faulty means that none of the components or exactly one of the components is faulty. None are faulty is the same as all 10 are in working order which was evaluated in part (b) (i) above. One is faulty is same as 9 are in working order which was evaluated in part (b) (ii) above. We have

$$P(\text{at most one is faulty}) = 0.7374 + 0.2052 = 0.9426$$