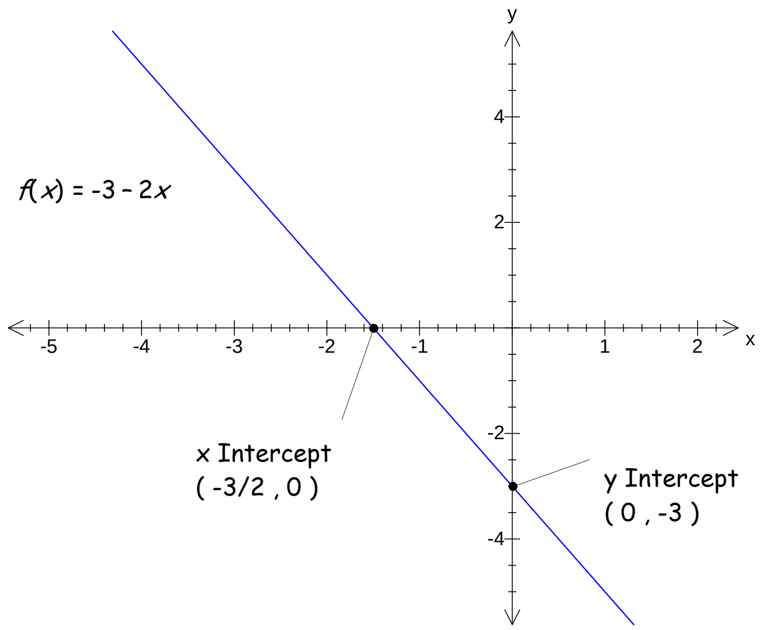
Complete Solutions to Examination Questions 3

1. The graph of $f(x) = -3 - 2x$ is a straight line with gradient equal to -2 and y intercept equal to -3. *Where does the graph cross the x axis?*

Where $f(x) = 0$ which means $-3 - 2x = 0$ and this implies that $x = -\frac{3}{2}$:



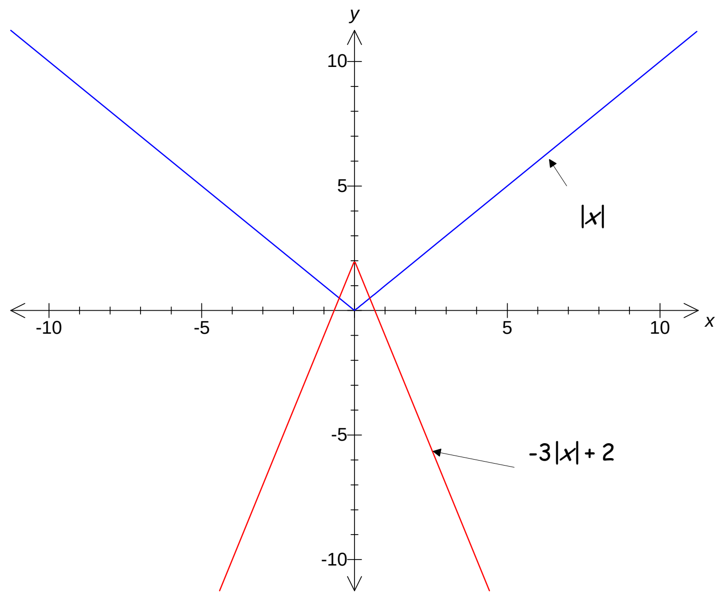
The domain and range is the set of all real numbers.

2. (a) The graph of $y = -3|x| + 2$ is similar to the graph of $y = |x|$. We can obtain the graph of $y = -3|x| + 2$ from $y = |x|$ by carrying out the following:

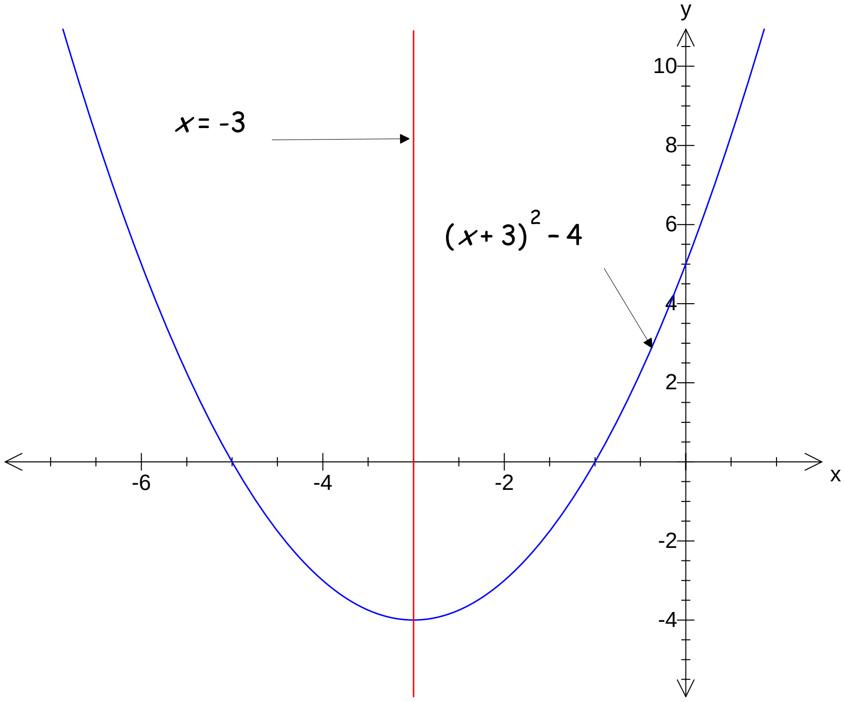
(i) inverting because of the minus sign

(ii) squashing horizontally by a factor of 3 because of the 3 in front of $|x|$

(iii) shifted up by 2 units because of the $+2$



(b) The graph of $f(x) = (x+3)^2 - 4$ is the x^2 graph but shifted to the left by 3 units and down by 4 units:



The axis of symmetry is $x = -3$.

3. The graph of $f(x) = 2 - \sqrt{x+1}$ is basically the graph of $\sqrt{x}$ with some adjustments.

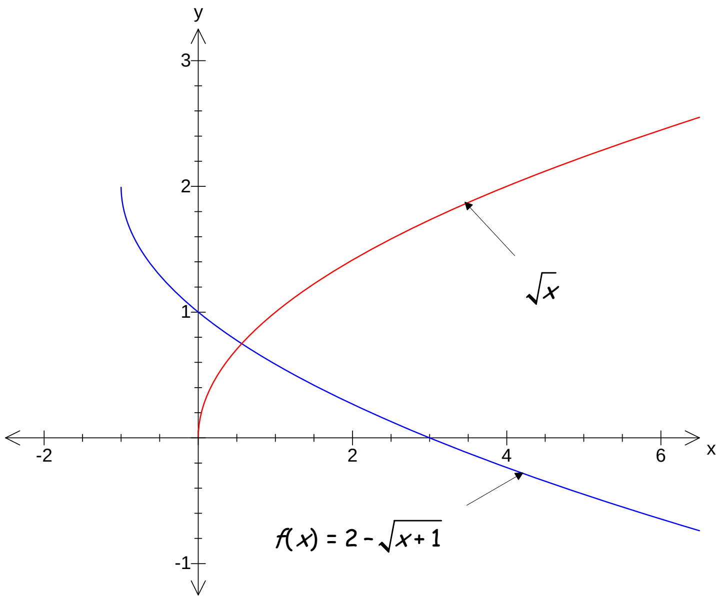
How does the graph of $\sqrt{x}$ *change in order to give* $f(x) = 2 - \sqrt{x+1}$ *?*

(i) $\sqrt{x+1}$ shifts the graph of $\sqrt{x}$ to the left by 1 unit.

(ii) $-\sqrt{x+1}$ reflects the graph in the *x* axis.

(iii) $f(x) = 2 - \sqrt{x+1}$ shifts the graph up by 2 units.

By combining these we have



By examining the graph, we have that the domain of $f(x)$ is $x \geq -1$ and range is $f(x) \leq 2$.

4. The evaluation of $(R-C)(600)$ is given by substituting $x=600$ into $R(x)=85x$ minus $C(x)=65x+3500$:

$$(R-C)(x)=85x-(65x+3500)=20x-3500$$
$$(R-C)(600)=(20\times 600)-3\,500=12\,000-3\,500=8\,500$$

$(R-C)(600)=8\,500$ means the company makes a profit of \$8 500 because we have evaluated revenue minus costs.

5. *How do we evaluate* $\dfrac{f(x+h)-f(x)}{h}$ *for* $f(x)=\dfrac{1}{3x}$ *?*

In this case $f(x)=\dfrac{1}{3x}$ and $f(x+h)=\dfrac{1}{3(x+h)}$. Substituting these into $f(x+h)-f(x)$:

$$\begin{aligned} f(x+h)-f(x) &= \frac{1}{3(x+h)}-\frac{1}{3x} \\ &= \frac{1}{3x+3h}-\frac{1}{3x} \\ &= \frac{3x-(3x+3h)}{3x(3x+3h)}=-\frac{3h}{9(x^2-xh)}=-\frac{h}{3(x^2-xh)} \end{aligned}$$

Dividing this result by h yields:

$$\frac{f(x+h)-f(x)}{h}=-\frac{\cancel{h}}{3(x^2-xh)\cancel{h}}=-\frac{1}{3(x^2-xh)}$$

6. a. The function $(g\circ f)(-1)$ means $g(f(-1))$ given $f(x)=\dfrac{1-2x}{3x}$ and $g(x)=3x^2-9x$:

$$\begin{aligned} g(f(-1)) &= g\left(\frac{1-2(-1)}{3(-1)}\right) && \left[\text{Because } f(x)=\frac{1-2x}{3x}\right] \\ &= g\left(\frac{3}{-3}\right)=g(-1) \\ &= 3(-1)^2-9(-1)=12 && \left[\text{Because } g(x)=3x^2-9x\right] \end{aligned}$$

b. Similarly we have

$$\begin{aligned} (f\circ g)(2) &= f(g(2)) \\ &= f\left(3(2)^2-9(2)\right) && \left[\text{Because } g(x)=3x^2-9x\right] \\ &= f(-6) \\ &= \frac{1-2(-6)}{3(-6)}=-\frac{13}{18} && \left[\text{Because } f(x)=\frac{1-2x}{3x}\right] \end{aligned}$$

7. Let y equal $f(x)=\dfrac{x-5}{2x+3}$ and then transpose to make x the subject of the formula:

$$y = \frac{x-5}{2x+3}$$
$$y(2x+3) = x-5$$
$$2xy+3y = x-5$$
$$3y+5 = x-2xy = x(1-2y)$$
$$\frac{3y+5}{1-2y} = x$$

Hence the inverse function is given by $f^{-1}(x) = \frac{3x+5}{1-2x}$ provided $x \neq \frac{1}{2}$.

8. (a) We have

$$\lim_{x\to -2} \frac{x^2-x-6}{x+2} = \lim_{x\to -2} \frac{(x+2)(x-3)}{x+2} \underbrace{=}_{\substack{\text{Cancelling}\\ x+2}} \lim_{x\to -2}(x-3) = -5$$

(b) *How do we find* $\lim_{x\to 8} \frac{\sqrt{x+1}-3}{x-8}$ *?*

Multiply numerator and denominator by $\sqrt{x+1}+3$:

$$\lim_{x\to 8} \frac{\sqrt{x+1}-3}{x-8} = \lim_{x\to 8} \frac{\left(\sqrt{x+1}-3\right)\left(\sqrt{x+1}+3\right)}{(x-8)\left(\sqrt{x+1}+3\right)}$$
$$= \lim_{x\to 8} \frac{x+1-9}{(x-8)\left(\sqrt{x+1}+3\right)}$$
$$= \lim_{x\to 8} \frac{x-8}{(x-8)\left(\sqrt{x+1}+3\right)}$$
$$\underbrace{=}_{\substack{\text{Cancelling}\\ x-8}} \lim_{x\to 8} \frac{1}{\left(\sqrt{x+1}+3\right)} = \frac{1}{6}$$

(c) We need to evaluate $\lim_{x\to\infty} \frac{1-12x}{4x+5}$. Writing $1-12x = 16-3(4x+5)$ we have

$$\lim_{x\to\infty} \frac{1-12x}{4x+5} = \lim_{x\to\infty} \frac{16-3(4x+5)}{4x+5}$$
$$= \lim_{x\to\infty}\left(\frac{16}{4x+5}\right) - 3\lim_{x\to\infty}\left(\frac{4x+5}{4x+5}\right)$$
$$= 0-3 = -3$$

9. *What does* $fg(x)$ *mean?*

This means first find $g(x)$ and then find *f* of this with the argument being $g(x)$:

$$f\left(g\left(x\right)\right)=f\left(\frac{3}{x}+2\right)$$

$$=4-\left(\frac{3}{x}+2\right)^2 \qquad \left[\text{Because } f\left(x\right)=4-x^2\right]$$

$$=4-\left(\frac{3+2x}{x}\right)^2$$

$$=4-\frac{9+2(3)(2x)+(2x)^2}{x^2}=\frac{4x^2-9-12x-4x^2}{x^2}=-\frac{9+12x}{x^2}$$

Similarly we have

$$f\left(f\left(x\right)\right)=f\left(4-x^2\right)$$

$$=4-\left(4-x^2\right)^2$$

$$=4-\underbrace{\left(16-2\left(4\right)x^2+\left(x^2\right)^2\right)}_{\text{Using } (a-b)^2=a^2-2ab+b^2}$$

$$=4-16+8x^2-x^4=-x^4+8x^2-12$$

What does $g^{-1}\left(x\right)$ *mean?*

$g^{-1}\left(x\right)$ is the inverse function of *g*. *How do we find the inverse function?*

Let $y=g\left(x\right)$ and then make *x* the subject of this formula. Thus we have

$$y=\frac{3}{x}+2$$

$$yx=3+2x$$

$$yx-2x=3$$

$$x\left(y-2\right)=3 \quad \Rightarrow \quad x=\frac{3}{y-2}$$

Therefore $g^{-1}\left(x\right)=\dfrac{3}{x-2}$ provided $x\neq 2$.

10. We need to find the inverse of $f\left(x\right)=2-x^2$, $x\leq 0$. Let $y=f\left(x\right)$ and then transpose to
make *x* the subject. We have

$$y=2-x^2$$

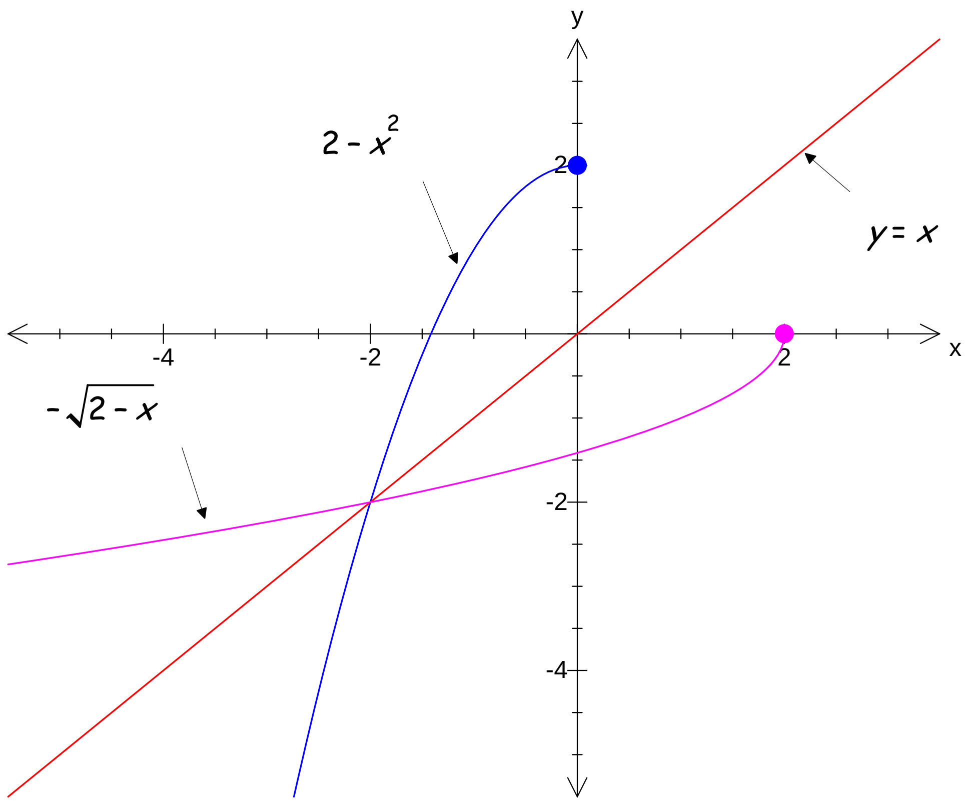
$$x^2=2-y$$

$$x=\pm\sqrt{2-y}$$

Since we are given that $x\leq 0$ therefore $x=-\sqrt{2-y}$ which means $f^{-1}\left(x\right)=-\sqrt{2-x}$ provided $x\leq 2$.

The graph of $f\left(x\right)=2-x^2$ is the inverted graph of x^2 which has been shifted up by 2 units.

We can sketch $f^{-1}\left(x\right)=-\sqrt{2-x}$ by reflecting the graph of $f\left(x\right)=2-x^2$ in the line $y=x$:



11. Substituting $t=0$ into the given formula $p(t)=5+2t-t^2$ we have

$$p(0)=5+2(0)-0^2=5$$

Substituting $t=4$ gives

$$p(4)=5+2(4)-4^2=-3$$

Since $p(t)$ goes from positive to negative (goes from west to east) between $t=0$ and $t=4$ therefore Jennifer must have visited the Creamery during these hours.

12. (a) We need to evaluate $\lim\limits_{x\to(-1)}\dfrac{x^4-3x^2-8}{x^2-\sqrt{4x+8}}$. Since the denominator is **not** zero at $x=-1$ therefore we can substitute this into the above:

$$\lim_{x\to(-1)}\frac{x^4-3x^2-8}{x^2-\sqrt{4x+8}}=\frac{(-1)^4-3(-1)^2-8}{(-1)^2-\sqrt{4(-1)+8}}=\frac{-10}{-1}=10$$

(b) *How do we evaluate* $\lim\limits_{x\to(-2)}\dfrac{x^2-3x-10}{x^2+x-2}$ *?*

We can factorize the numerator and denominator as

$$\frac{x^2-3x-10}{x^2+x-2}=\frac{(x-5)(x+2)}{(x+2)(x-1)}\underset{\substack{\text{Cancelling}\\ x+2}}{=}\frac{x-5}{x-1}$$

Hence we have $\lim\limits_{x\to(-2)}\dfrac{x^2-3x-10}{x^2+x-2}=\lim\limits_{x\to(-2)}\dfrac{x-5}{x-1}=\dfrac{-7}{-3}=\dfrac{7}{3}$.

(c) We need to split the numerator of $\dfrac{h+\sqrt{4+h}-2}{h}$ as:

$$\lim_{h\to 0}\frac{h+\sqrt{4+h}-2}{h}=\lim_{h\to 0}\left(\frac{\not{h}}{\not{h}}\right)+\lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\right) \qquad (*)$$

How do we evaluate $\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$ *?*

Multiply numerator and denominator by $\sqrt{4+h}+2$:

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}=\lim_{h\to 0}\frac{\left(\sqrt{4+h}-2\right)\left(\sqrt{4+h}+2\right)}{h\left(\sqrt{4+h}+2\right)}$$

$$=\lim_{h\to 0}\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)}$$

$$=\lim_{h\to 0}\frac{\not{h}}{\not{h}\left(\sqrt{4+h}+2\right)}=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}=\frac{1}{4}$$

Substituting this $\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}=\frac{1}{4}$ into (*) yields:

$$\lim_{h\to 0}\frac{h+\sqrt{4+h}-2}{h}=\lim_{h\to 0}\left(\frac{\not{h}}{\not{h}}\right)+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4}$$