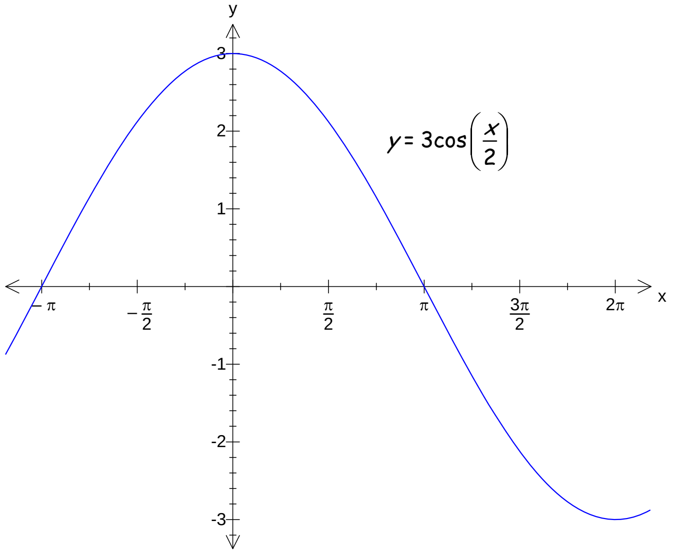
Complete Solutions to Examination Questions 4

1. (a) *How do we sketch the graph of* $y = 3\cos\left(\frac{x}{2}\right)$?

Well it is a cos graph with amplitude of 3 and only half a waveform between 0 and 2π because the cos argument is $x/2$:



(b) Using the waveform section in the textbook with reference to (4.28), (4.29) and (4.30) we have the following:

The period T is the time taken to complete one cycle, that is $T = \frac{2\pi}{1/2} = 4\pi$. The frequency f is defined as $f = \frac{1}{T} = \frac{1}{4\pi}$ Hz. The amplitude of the given function is 3.

2. Taking inverse cos of both sides gives $3\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$. By formula

(4.17) The general solution of $\cos(\theta) = R$ is $\theta = (360 \times n)° \pm \alpha$ where $\alpha = \cos^{-1}(R)$

the general solution is given by

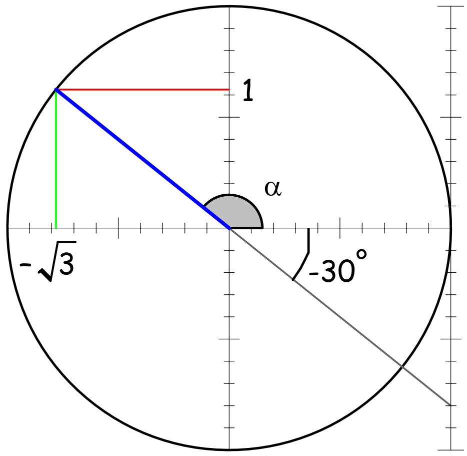
$$3\theta = (360 \times n)° \pm 120°$$

$$\theta \underbrace{=}_{\text{Dividing by 3}} \frac{(360 \times n)° \pm 120°}{3} = \frac{(360 \times n)°}{3} \pm \frac{120°}{3} = (120n)° \pm 40°$$

3. *How do we convert* $\sin x - \sqrt{3}\cos x$ *into* $r\cos(x - \alpha)$?

Finding the angle α is difficult but to determine r is a straightforward application of Pythagoras, that is $r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$.

Plotting the quadrant where $\sin x - \sqrt{3}\cos x$ lies:



By using a calculator or TABLE 1 we have

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30°$$

Clearly $-30°$ is the **wrong** quadrant. Hence

$$\alpha = 180° - 30° = 150°$$

Substituting $r = 2$ and $\alpha = 150°$ we have

$$\sin x - \sqrt{3}\cos x = 2\cos\left(x - 150°\right)$$

4. Cross-multiplying the Left Hand Side of $\dfrac{\sin A}{\sin B} + \dfrac{\cos A}{\cos B} = 2\dfrac{\sin\left(A+B\right)}{\sin 2B}$ gives

$$\frac{\sin A\cos B + \cos A\sin B}{\sin B\cos B} = \frac{\sin\left(A+B\right)}{\sin B\cos B} \qquad \left[\begin{array}{l}\text{Because} \\ \sin A\cos B + \cos A\sin B = \sin\left(A+B\right)\end{array}\right]$$

$$= \frac{\sin\left(A+B\right)}{\frac{1}{2}\sin 2B} \qquad \left[\text{Because } \sin\left(2B\right) = 2\sin B\cos B\right]$$

$$= \frac{2\sin\left(A+B\right)}{\sin 2B} \qquad \left[\text{Because } \frac{1}{1/2} = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2\right]$$

How do we evaluate $\dfrac{\sin 75°}{\sin 60°}$?

We need to write the $75°$ and $60°$ in terms of well known trigonometric ratios such as those in TABLE 1. We can write

$$75° = 45° + 30° \text{ and } 60° = 2\left(30°\right)$$

Substituting these into the above we have

$$\frac{\sin 75°}{\sin 60°} = \frac{\sin\left(45° + 30°\right)}{\sin\left(2\left[30°\right]\right)}$$

We can evaluate this by using the above proven result $\dfrac{\sin A}{\sin B} + \dfrac{\cos A}{\cos B} = 2\dfrac{\sin\left(A+B\right)}{\sin 2B}$.

Dividing this by 2 we have

$$\frac{\sin\left(A+B\right)}{\sin 2B} = \frac{1}{2}\left[\frac{\sin A}{\sin B} + \frac{\cos A}{\cos B}\right] \qquad (*)$$

Substituting $A=45°$ and $B=30°$ into (*) gives

$$\frac{\sin 75°}{\sin 60°} \underbrace{=}_{\text{From above}} \frac{\sin\left(45°+30°\right)}{\sin\left(2\left[30°\right]\right)} = \frac{1}{2}\left[\frac{\sin 45°}{\sin 30°}+\frac{\cos 45°}{\cos 30°}\right]$$

$$= \frac{1}{2}\left[\frac{1/\sqrt{2}}{1/2}+\frac{1/\sqrt{2}}{\sqrt{3}/2}\right] \qquad \left[\text{Because } \sin\left(45°\right)=\cos\left(45°\right)=\frac{1}{\sqrt{2}}\right]$$

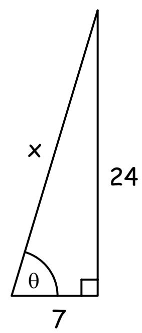
$$= \frac{1}{2}\left[\left(\frac{1}{\sqrt{2}}\div\frac{1}{2}\right)+\left(\frac{1}{\sqrt{2}}\div\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{\sqrt{2}}\times\frac{2}{1}\right)+\left(\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{3}}\right)\right]$$

$$\underbrace{=}_{\text{Taking out common factor 2}} \frac{1}{\cancel{2}}\cancel{2}\left[\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{3}}\right)\right] \underbrace{=}_{\text{Taking out common factor } 1/\sqrt{2}} \frac{1}{\sqrt{2}}\left[1+\frac{1}{\sqrt{3}}\right]$$

5. *What does* $\cos\left(\arctan\left(\frac{24}{7}\right)\right)$ *mean?*

Remember arctan is $\tan^{-1}$, that is $\arctan\left(\frac{24}{7}\right)=\tan^{-1}\left(\frac{24}{7}\right)$. Let θ be the angle such that $\theta=\tan^{-1}\left(\frac{24}{7}\right)$ therefore $\tan\left(\theta\right)=\frac{24}{7}$. We have



By Pythagoras $x=\sqrt{24^2+7^2}=25$. Therefore $\cos\left(\theta\right)=\frac{\text{adj}}{\text{hyp}}=\frac{7}{25}$:

$$\cos\left(\arctan\left(\frac{24}{7}\right)\right)=\cos\left(\tan^{-1}\left(\frac{24}{7}\right)\right)=\cos\left(\theta\right)=\frac{7}{25}$$

6. We need to find the values of t which satisfy $\cos\left(2t\right)-2\sin^2\left(t\right)=0$. *How?*

We use a trig identity but which one?

Use $\cos\left(2t\right)=1-2\sin^2\left(t\right)$. Thus substituting this $\cos\left(2t\right)=1-2\sin^2\left(t\right)$ into the given equation $\cos\left(2t\right)-2\sin^2\left(t\right)=0$ yields:

$$1-2\sin^2\left(t\right)-2\sin^2\left(t\right)=1-4\sin^2\left(t\right)=0$$

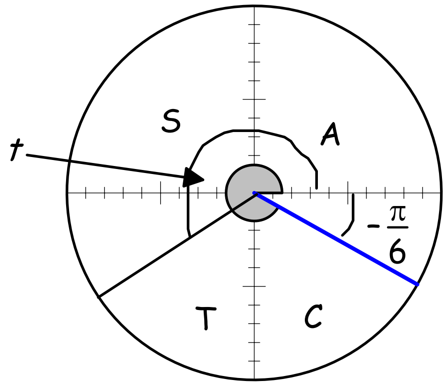
$$\sin^2\left(t\right)=\frac{1}{4} \qquad \left[\text{Transposing}\right]$$

Taking the square root gives

$$\sin(t) = -\frac{1}{2},\ \frac{1}{2}$$

What are the values of t where $\sin(t) = -\frac{1}{2}$ *?*

Using our calculator or by the table of values in the text we have $t = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. The domain of t is between 0 and 2π therefore $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. We can use CAST to find the other angle:



$\sin(t) = -\frac{1}{2}$ (Negative) in the quadrants C and T.

The other angle t is given by $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$. *Do we have any other values of t?*

Yes where $\sin(t) = +\frac{1}{2}$. In this case t lies in the quadrants A and S because we have positive value of sin. Hence $t = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Collecting our t values we have $t = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

7. We can write $\frac{13\pi}{12}$ as $\frac{3\pi}{4} + \frac{\pi}{3} = \frac{13\pi}{12}$. We have

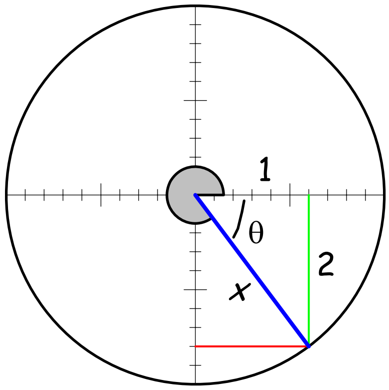
$$\cos\left(\frac{13\pi}{12}\right) = \cos\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$

$$= \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \qquad \left[\begin{array}{l}\text{Using } \cos(A+B) = \\ \cos(A)\cos(B) - \sin(A)\sin(B)\end{array}\right]$$

$$= -\frac{1}{\sqrt{2}}\frac{1}{2} - \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} \qquad \left[\begin{array}{l}\text{Because } \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}},\ \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}, \\ \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\end{array}\right]$$

$$\underbrace{=}_{\text{Taking Out a Common Factor}} -\frac{1}{\sqrt{2}}\frac{1}{2}\left[1 + \sqrt{3}\right]$$

8. Since $\tan(\theta) = -2$ is in the fourth quadrant we have:



The hypotenuse $x = \sqrt{2^2 + 1^2} = \sqrt{5}$. We have

$$\sin(\theta) = -\frac{2}{\sqrt{5}} \text{ and } \cos(\theta) = \frac{1}{\sqrt{5}}$$

Using the double angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$ we have

$$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

Re-arranging this gives

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left[1 - \cos(\theta)\right]$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left[1 - \cos(\theta)\right]} \qquad \left[\text{Taking Square Root}\right]$$

Substituting the above result $\cos(\theta) = \frac{1}{\sqrt{5}}$ we have

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left[1 - \cos(\theta)\right]}$$

$$= \pm\sqrt{\frac{1}{2}\left[1 - \frac{1}{\sqrt{5}}\right]}$$

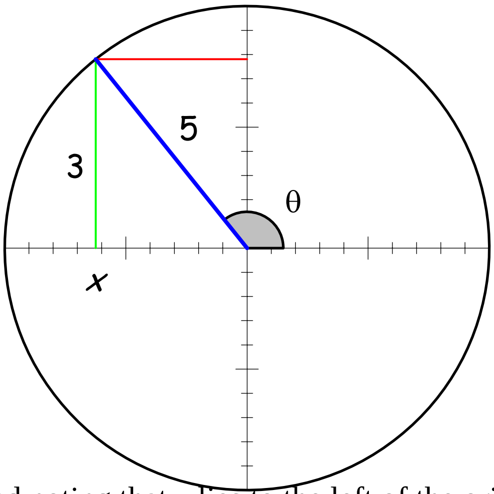
We are given that θ is in the fourth quadrant which means that $\frac{3\pi}{2} < \theta < 2\pi$. Dividing this by 2:

$$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$$

Therefore $\frac{\theta}{2}$ is in the second quadrant where the sin value is positive. Removing the negative square root we have

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}\left[1 - \frac{1}{\sqrt{5}}\right]}$$

9. Since θ is in the second quadrant such that $\sin(\theta)=\frac{3}{5}$, we have:



By applying Pythagoras and noting that *x* lies to the left of the origin we have

$$x=-\sqrt{5^2-3^2}=-4$$

Remember the negative only signifies length *x* is to the left of the origin.
Therefore using the definition of cosine and tangent

$$\cos(\theta)=-\frac{4}{5},\quad \tan(\theta)=-\frac{3}{4}$$

What other trigonometric ratios do we need to find?

$$\operatorname{cosec}(\theta)=\frac{1}{\sin(\theta)}=\frac{1}{3/5}=\frac{5}{3}$$

$$\sec(\theta)=\frac{1}{\cos(\theta)}=\frac{1}{-4/5}=-\frac{5}{4}$$

$$\cot(\theta)=\frac{1}{\tan(\theta)}=\frac{1}{-3/4}=-\frac{4}{3}$$

10. *How do we verify* $\frac{1}{\cos(x)\csc^2(x)}=\sec(x)-\cos(x)$*?*

Note that $\csc(x)=\operatorname{cosec}(x)=\frac{1}{\sin(x)}$ which means that $\frac{1}{\csc(x)}=\frac{1}{\operatorname{cosec}(x)}=\sin(x)$.

Substituting this into the given equation we have

$$\frac{1}{\cos(x)\csc^2(x)}=\frac{\sin^2(x)}{\cos(x)} \qquad \left[\text{Because } \frac{1}{\csc^2(x)}=\sin^2(x)\right]$$

$$=\frac{1-\cos^2(x)}{\cos(x)}=\frac{1}{\cos(x)}-\frac{\cos^2(x)}{\cos(x)}$$

$$=\sec(x)-\cos(x) \qquad \left[\text{Because } \frac{1}{\cos(x)}=\sec(x)\right]$$

11. In the textbook we have only converted from a combination of sine and cosine to

a cosine only waveform. In this case we will use the otherwise option, that is convert the Left Hand Side of $\sin x + 3\cos x = 2.4$ into $R\cos(x-\alpha)$. By

(4.75) $\quad a\cos(\theta) + b\sin(\theta) = R\cos(\theta - \alpha)$ where $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left(\dfrac{b}{a}\right)$

with $a = 3$ and $b = 1$ we have

$$R = \sqrt{3^2+1^2} = \sqrt{10}$$

$$\alpha = \tan^{-1}\left(\frac{1}{3}\right) = 18.43° \quad \text{[By calculator]}$$

Solving the given equation we have

$$\sin x + 3\cos x = \sqrt{10}\cos(x - 18.43°) = 2.4$$

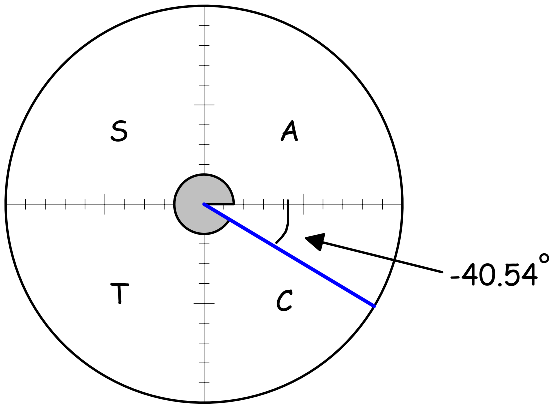
Dividing through by $\sqrt{10}$ and taking inverse cosine we have

$$\cos(x - 18.43°) = \frac{2.4}{\sqrt{10}} = 0.76$$

$$x - 18.43° = \cos^{-1}(0.76) = 40.54°$$

$$x = 40.54° + 18.43° = 58.97°$$

Hence $x = 58.97°$. *Are there any other angles?*
Yes because cos is positive in the fourth quadrant as well.



The angle shown is $360° - \cos^{-1}(0.76) = 360° - 40.54° = 319.46°$. Using this angle we have

$$x - 18.43° = 319.46°$$

$$x = 319.46° + 18.43° = 337.89°$$

The angles are $x = 58.97°,\ 337.89°$.

12. (a) *How do we show* $\sin\left(x - \dfrac{3\pi}{2}\right) = \cos(x)$ *?*

Apply the identity $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ with $A = x$ and $B = \dfrac{3\pi}{2}$:

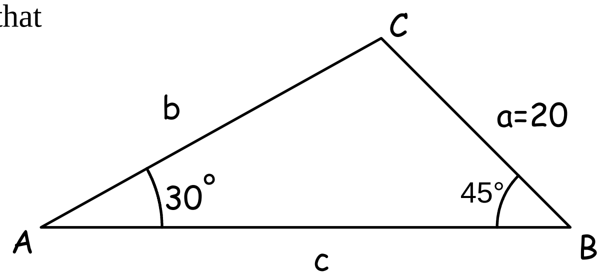
$$\sin\left(x - \frac{3\pi}{2}\right) = \sin(x)\underbrace{\cos\left(\frac{3\pi}{2}\right)}_{=0} - \cos(x)\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1}$$

$$= 0 - \cos(x)(-1) = \cos(x)$$

(b) Writing $\sec(x) = \dfrac{1}{\cos(x)}$ into the LHS of the given equation $\sec x - \cos x = \sin x \tan x$:

$$\begin{aligned}\sec(x) - \cos(x) &= \frac{1}{\cos(x)} - \cos(x)\\ &= \frac{1-\cos^2(x)}{\cos(x)}\\ &= \frac{\sin^2(x)}{\cos(x)} && \left[\text{Because } \cos^2(x)+\sin^2(x)=1\right]\\ &= \sin(x)\frac{\sin(x)}{\cos(x)} = \sin(x)\tan(x) && \left[\text{Because } \frac{\sin}{\cos} = \tan\right]\end{aligned}$$

13. We are given that



Which rule do we use to find the length b?
We can apply the sine rule because we have two angles and one length.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} \qquad (*)$$

Substituting $a = 20$, $A = 30°$ and $B = 45°$ into (*) we have

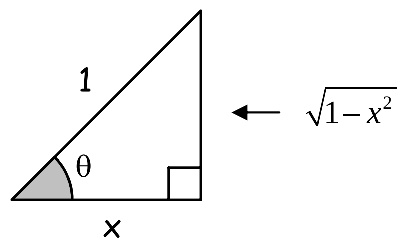
$$\frac{20}{\sin(30°)} = \frac{b}{\sin(45°)}$$

Transposing this gives $b = \dfrac{20}{\sin(30°)} \times \sin(45°) = 28.28$.

14. Using the identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = \arctan 1$ and $B = \arccos x$ we have

$$\begin{aligned}\cos(\arctan 1 + \arccos x) &= \cos(A)\cos(B) - \sin(A)\sin(B)\\ &= \cos(\arctan 1)\cos(\arccos x) - \sin(\arctan 1)\sin(\arccos x) \qquad (\dagger)\end{aligned}$$

Using our table of values we have $\arctan 1 = 45°$. Let θ be the angle such that $\cos(\theta) = x$ then $\theta = \arccos(x)$. Drawing a right-angled triangle with hypotenuse $= 1$ and adjacent $= x$ because $\cos(\theta) = x$:



Using the sine definition (4.1) we have $\sin(\theta)=\dfrac{\sqrt{1-x^2}}{1}=\sqrt{1-x^2}$.

Substituting $\arctan 1=45°$ and $\theta=\arccos(x)$ into $(\dagger)$ yields:

$$\cos(\arctan 1+\arccos x)=\cos(\arctan 1)\cos(\arccos x)-\sin(\arctan 1)\sin(\arccos x)$$

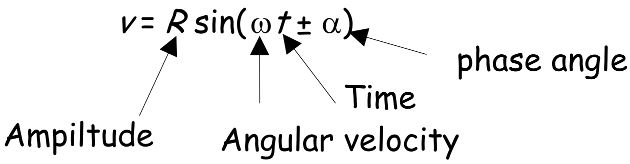
$$=\cos(45°)\underbrace{\cos(\theta)}_{=x}-\sin(45°)\underbrace{\sin(\theta)}_{=\sqrt{1-x^2}}$$

$$=\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}\sqrt{1-x^2} \qquad \left[\text{Because } \cos(45°)=\sin(45°)=\frac{1}{\sqrt{2}}\right]$$

$$=\frac{1}{\sqrt{2}}\left[x-\sqrt{1-x^2}\right] \qquad \left[\text{Factorising } \frac{1}{\sqrt{2}}\right]$$

Therefore the expression in x is $\dfrac{1}{\sqrt{2}}\left[x-\sqrt{1-x^2}\right]$.

15. The given waveform is $v(t)=57\sin(67\pi t-0.26)$. Remember we know from the main text that



For $v(t)=57\sin(67\pi t-0.26)$ we have $R=57$ and $\omega=67\pi$. Using these values:

(i) Amplitude $=57$, period time $T=\dfrac{2\pi}{\omega}=\dfrac{2\cancel{\pi}}{67\cancel{\pi}}=\dfrac{2}{67}=29.85\times10^{-3}=29.85$ ms. Using $f=\dfrac{1}{T}$ we have frequency $f=\dfrac{1}{29.85\times10^{-3}}=33.5$ Hz. The phase is 0.26 rads. *How do we convert this into degrees?*

By applying (4.27) $x \text{ rads}=\left(\dfrac{x\times180}{\pi}\right)°$:

$$0.26=\frac{0.26\times180}{\pi}=15°$$

The phase angle in degrees is $15°$.

How do we find the phase time?

By using (4.33) $R\sin(\omega t-\alpha)$ lags $R\sin(\omega t)$ by $\dfrac{\alpha}{\omega}$. Hence

$$\text{Phase time}=\frac{0.26}{67\pi}=1.235\times10^{-3}=1.235\text{ ms}$$

Hence the phase time is 1.235 ms.

(ii) Substitute $t=0$ into the given equation $v(t)=57\sin(67\pi t-0.26)$:

$$v(0)=57\sin(67\pi(0)-0.26)=57\sin(-0.26)=-14.65$$

At $t=0$ the voltage is -14.65 volts.

(iii) Substituting $t=8\text{ ms}=8\times10^{-3}$ into $v(t)=57\sin(67\pi t-0.26)$ gives

$$v\left(8\times10^{-3}\right)=57\sin\left(67\pi\left(8\times10^{-3}\right)-0.26\right)$$
$$=57\sin\left(1.42\right)=56.35$$

The voltage at $t=8$ ms is 56.35 volts.

(iv) *How do we find the time when the voltage is first a maximum?*

This occurs when $\sin\left(67\pi t-0.26\right)=1$ because the maximum sine value is 1. Taking inverse sine gives

$$67\pi t-0.26=\sin^{-1}\left(1\right)=\frac{\pi}{2}$$
$$67\pi t=\frac{\pi}{2}+0.26$$
$$t=\frac{\cancel{\pi}}{2\left(67\cancel{\pi}\right)}+\frac{0.26}{67\pi}=8.70\times10^{-3}$$

The voltage is first a maximum at time $t=8.70$ ms.

(v) *When does* $v=40$ *volts?*

Substituting $v\left(t\right)=40$ into $v\left(t\right)=57\sin\left(67\pi t-0.26\right)$:

$$57\sin\left(67\pi t-0.26\right)=40$$
$$\sin\left(67\pi t-0.26\right)=\frac{40}{57}=0.70$$

Taking inverse sin gives

$$67\pi t-0.26=\sin^{-1}\left(0.70\right)=0.775$$

Transposing to make t the subject gives

$$t=\frac{0.775+0.26}{67\pi}=4.92\times10^{-3}$$

The waveform $v\left(t\right)$ reaches 40volts at $t=4.92$ ms.

We can sketch $v\left(t\right)$ over one cycle by using the above evaluations: