Complete Solutions to Examination Questions 6

1. (i) To differentiate the given function $y = 5x^{-3} + \frac{7}{x} + \frac{2x^6}{3} - 9x + 26$ we need to rewrite the term $\frac{7}{x} = 7x^{-1}$. Differentiating

$$y = 5x^{-3} + 7x^{-1} + \frac{2x^6}{3} - 9x + 26$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -15x^{-4} - 7x^{-2} + 6\left(\frac{2x^5}{3}\right) - 9$$

$$= -15x^{-4} - 7x^{-2} + 4x^5 - 9 \qquad \left[\text{Simplifying } 6\left(\frac{2x^5}{3}\right) = 2\left(2x^5\right) = 4x^5\right]$$

(ii) *How do we differentiate* $y = 2x\ln\left(\frac{2}{x}\right)$*?*

Since we have a product therefore we use the product rule which is

(6.31) $$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u'v + uv'$$

Using this formula and basic rules of logs we have

$$u = 2x \qquad v = \ln\left(\frac{2}{x}\right) = \ln(2) - \ln(x) \qquad \left[\text{Using } \ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)\right]$$

$$u' = 2 \qquad v' = 0 - \frac{1}{x} = -\frac{1}{x} \qquad \left[\text{Note } \ln(2) = 0.693 \text{ is a constant}\right]$$

Substituting these into the product rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2x\ln\left(\frac{2}{x}\right)\right] = u'v + uv'$$

$$= 2\ln\left(\frac{2}{x}\right) + 2\cancel{x}\left(-\frac{1}{\cancel{x}}\right)$$

$$\underbrace{=}_{\text{Cancelling}} 2\ln\left(\frac{2}{x}\right) - 2 = 2\left[\ln\left(\frac{2}{x}\right) - 1\right] \qquad \left[\text{Taking out } 2\right]$$

(iii) *How do we differentiate* $y = \frac{4e^{-2x}}{\sin(5x)}$*?*

We have a quotient function therefore we use the quotient rule:

(6.32) $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$$

Let

$$u = 4e^{-2x} \qquad v = \sin(5x)$$

$$u' = -8e^{-2x} \qquad v' = 5\cos(5x) \qquad \left[\text{Differentiating}\right]$$

Substituting these into the quotient rule formula (multiplying the diagonals goes on top):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{4e^{-2x}}{\sin(5x)}\right]=\frac{u'v-uv'}{v^2}$$

$$=\frac{-8e^{-2x}\sin(5x)-4e^{-2x}5\cos(5x)}{\left[\sin(5x)\right]^2}$$

$$=\frac{-4e^{-2x}\left[2\sin(5x)+5\cos(5x)\right]}{\sin^2(5x)}\qquad\left[\begin{array}{l}\text{Taking out common factor}\\ -4e^{-2x}\ \text{ on numerator}\end{array}\right]$$

(iv) We will not find a derivative formula for the given function $y=\sqrt{1+5x^2-4x^3}$. We need to rewrite this by using the rules of indices, $\sqrt{a}=a^{1/2}$:

$$y=\sqrt{1+5x^2-4x^3}=\left(1+5x^2-4x^3\right)^{1/2}$$

How do we differentiate this $y=\left(1+5x^2-4x^3\right)^{1/2}$ *?*

Use the chain rule directly, that is $\frac{\mathrm{d}}{\mathrm{d}x}\left(u^n\right)=nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$ with $u=1+5x^2-4x^3$:

$$y=\left(1+5x^2-4x^3\right)^{1/2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(1+5x^2-4x^3\right)^{-1/2}\left[10x-12x^2\right]$$

$$\underbrace{=}_{\text{Using } a^{-1/2}=\frac{1}{\sqrt{a}}}\frac{1}{2}\frac{10x-12x^2}{\sqrt{1+5x^2-4x^3}}=\frac{1}{\cancel{2}}\frac{\cancel{2}x\left[5-6x\right]}{\sqrt{1+5x^2-4x^3}}\underbrace{=}_{\text{Cancelling 2's}}\frac{x\left[5-6x\right]}{\sqrt{1+5x^2-4x^3}}$$

2. *How do we find the gradient of* $v=2\sin(t)+3t^2$ *at* $t=2$ *?*

Differentiate $v=2\sin(t)+3t^2$ with respect to t and then substitute $t=2$:

$$\frac{\mathrm{d}v}{\mathrm{d}t}=2\cos(t)+6t$$

Putting $t=2$ gives $2\cos(2)+(6\times 2)=11.17$.

3. Taking logs of both sides of the given equation $y=\frac{e^x\left(x^2+1\right)}{\sin(x)}$ we have

$$\ln(y)=\ln\left[\frac{e^x\left(x^2+1\right)}{\sin(x)}\right]$$

$$=\ln\left[e^x\left(x^2+1\right)\right]-\ln\left[\sin(x)\right]\qquad\left[\text{Applying }\ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)\right]$$

$$=\ln\left[e^x\right]+\ln\left(x^2+1\right)-\ln\left[\sin(x)\right]\qquad\left[\text{Using }\ln(AB)=\ln(A)+\ln(B)\right]$$

$$=x+\ln\left(x^2+1\right)-\ln\left[\sin(x)\right]\qquad\left[\text{Because ln and } e \text{ are inverses so }\ln\left[e^x\right]=x\right]$$

How do we differentiate this?

Using the chain rule directly on ln, that is $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(u)\right]=\frac{1}{u}\frac{\mathrm{du}}{\mathrm{d}x}$. Applying this to differentiate the above $\ln(y)=x+\ln\left(x^2+1\right)-\ln\left[\sin(x)\right]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(y)\right]=\frac{\mathrm{d}}{\mathrm{d}x}(x)+\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(x^2+1\right)\right]-\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left[\sin(x)\right]\right)$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=1+\frac{2x}{x^2+1}-\frac{\cos(x)}{\sin(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left[1+\frac{2x}{x^2+1}-\cot(x)\right] \qquad \left[\text{Remember } \frac{\cos(x)}{\sin(x)}=\cot(x)\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{e^x\left(x^2+1\right)}{\sin(x)}\left[1+\frac{2x}{x^2+1}-\cot(x)\right] \qquad \left[\begin{array}{l}\text{Substituting our given}\\ y=\dfrac{e^x\left(x^2+1\right)}{\sin(x)}\end{array}\right]$$

$$=\frac{e^x\left(x^2+1\right)}{\sin(x)}\left[\frac{x^2+1+2x-\left(x^2+1\right)\cot(x)}{x^2+1}\right] \qquad \left[\begin{array}{l}\text{Common}\\ \text{Denominator}\end{array}\right]$$

$$=\frac{e^x}{\sin(x)}\left[x^2+2x+1-\left(x^2+1\right)\cot(x)\right] \qquad \left[\text{Cancelling } x^2+1\right]$$

$$=\frac{e^x}{\sin(x)}\left[(x+1)^2-\left(x^2+1\right)\cot(x)\right] \qquad \left[\begin{array}{l}\text{Because}\\ x^2+2x+1=(x+1)^2\end{array}\right]$$

4. (i) We need to differentiate $f(x)=x^6-5x^3-x$:

$$f'(x)=6x^5-15x^2-1$$

(ii) *How do we differentiate* $w(\theta)=e^{2\theta}\cos(3\theta)$ *?*

Use the product rule, which is given by $\frac{\mathrm{d}}{\mathrm{d}x}[uv]=u'v+uv'$. We have

$$u=e^{2\theta} \qquad v=\cos(3\theta)$$
$$u'=2e^{2\theta} \qquad v'=-3\sin(3\theta)$$

Substituting these into the product rule formula (adding the multiplication of diagonals) gives

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{2\theta}\cos(3\theta)\right]=u'v+uv'$$

$$=2e^{2\theta}\cos(3\theta)+e^{2\theta}\left[-3\sin(3\theta)\right]$$

$$=e^{2\theta}\left[2\cos(3\theta)-3\sin(3\theta)\right] \qquad \left[\text{Taking out common factor } e^{2\theta}\right]$$

(iii) *How do we differentiate* $s(x)=\frac{\ln(2x)}{x^2+4}$ *?*

Since we have a quotient therefore we use the quotient rule $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right]=\frac{u'v-uv'}{v^2}$. Using this rule with

$$u = \ln(2x) \qquad v = x^2 + 4$$
$$u' = \frac{\cancel{2}}{\cancel{2}x} = \frac{1}{x} \qquad v' = 2x$$

Substituting these values into the above quotient rule gives

$$s'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln(2x)}{x^2+4}\right] = \frac{u'v - uv'}{v^2}$$
$$= \frac{\frac{1}{x}\left(x^2+4\right) - \ln(2x)2x}{\left(x^2+4\right)^2}$$
$$= \frac{\left(x^2+4\right) - x(2x)\ln(2x)}{x\left(x^2+4\right)^2} \qquad \left[\begin{array}{l}\text{Multiplying numerator}\\ \text{denominator by } x\end{array}\right]$$
$$= \frac{x^2+4-2x^2\ln(2x)}{x\left(x^2+4\right)^2}$$
$$= \frac{x^2\left(1-2\ln(2x)\right)+4}{x\left(x^2+4\right)^2}$$

Hence $s'(x) = \dfrac{x^2\left(1-2\ln(2x)\right)+4}{x\left(x^2+4\right)^2}$.

(iv) *How do we differentiate* $g(t) = e^{\sin(2t)}$ *?*

Use the chain rule directly, that is $\dfrac{\mathrm{d}}{\mathrm{dt}}\left[e^u\right] = e^u \dfrac{\mathrm{d}u}{\mathrm{dt}}$. Let $u = \sin(2t)$ then

$$\frac{\mathrm{du}}{\mathrm{dt}} = 2\cos(2t) \qquad \left[\text{Remember } \frac{\mathrm{d}}{\mathrm{dt}}\left[\sin(kt)\right] = k\cos(kt)\right]$$

We have $g'(t) = \dfrac{\mathrm{d}}{\mathrm{dt}}\left[e^{\sin(2t)}\right] = e^{\sin(2t)}2\cos(2t) = 2e^{\sin(2t)}\cos(2t)$.

(v) You will **not** find the derivative of $y(x) = \sqrt{x^3 - x}$ in any of the tables of chapter 6 because we have a square root sign. We need to rewrite this square root. *How?*
By using our rules of indices we have

$$y(x) = \sqrt{x^3-x} = \left(x^3-x\right)^{1/2} \qquad \left[\text{Remember } \sqrt{a} = a^{1/2}\right]$$

We can differentiate this by using the chain rule directly, that is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x^3-x\right)^{1/2}\right] = \frac{1}{2}\left(x^3-x\right)^{-1/2}\left[3x^2-1\right] \qquad \left[\text{Using } \frac{\mathrm{d}}{\mathrm{d}x}\left(u^n\right) = nu^{n-1}\frac{\mathrm{du}}{\mathrm{d}x}\right]$$
$$= \frac{3x^2-1}{2\sqrt{x^3-x}} \qquad \left[\text{Remember } a^{-1/2} = \frac{1}{\sqrt{a}}\right]$$

We have $y'(x) = \dfrac{3x^2-1}{2\sqrt{x^3-x}}$.

(vi) Need to differentiate $L(z) = \dfrac{2}{1+e^z}$. *How?*

Using the rules of indices we can write this as:

$$L(z) = 2\left(1+e^z\right)^{-1}$$

Applying $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(u^n\right) = nu^{n-1}\dfrac{\mathrm{d}u}{\mathrm{d}z}$:

$$L'(z) = -2\left(1+e^z\right)^{-2} e^z = \frac{-2e^z}{\left(1+e^z\right)^2} \qquad \text{[Differentiating and simplifying]}$$

The derivative is $\dfrac{-2e^z}{\left(1+e^z\right)^2}$.

5. (i) We need to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \dfrac{\pi}{4}$ for $y = \dfrac{\sin\left(x+\dfrac{\pi}{4}\right)}{1-\cos(x)}$. Since we have a quotient function therefore we apply the quotient rule:

(6.32) $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$$

Let

$$u = \sin\left(x+\frac{\pi}{4}\right) \qquad v = 1-\cos(x)$$

$$u' = \cos\left(x+\frac{\pi}{4}\right) \qquad v' = 0-\left[-\sin(x)\right] = \sin(x)$$

Substituting this into the quotient rule formula we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin\left(x+\dfrac{\pi}{4}\right)}{1-\cos(x)}\right] = \frac{u'v-uv'}{v^2}$$

$$= \frac{\cos\left(x+\dfrac{\pi}{4}\right)\left[1-\cos(x)\right] - \sin\left(x+\dfrac{\pi}{4}\right)\sin(x)}{\left[1-\cos(x)\right]^2}$$

$$= \frac{\cos\left(x+\dfrac{\pi}{4}\right) - \cos\left(x+\dfrac{\pi}{4}\right)\cos(x) - \sin\left(x+\dfrac{\pi}{4}\right)\sin(x)}{\left[1-\cos(x)\right]^2}$$

$$= \frac{\cos\left(x+\dfrac{\pi}{4}\right) - \left[\cos\left(x+\dfrac{\pi}{4}\right)\cos(x) + \sin\left(x+\dfrac{\pi}{4}\right)\sin(x)\right]}{\left[1-\cos(x)\right]^2} \qquad (\dagger)$$

Can we simplify this any further?

Yes we can use the trigonometric identity

(4.40) $$\cos(A)\cos(B)+\sin(A)\sin(B) = \cos(A-B)$$

on the square bracket term on the numerator. By substituting $A = x+\dfrac{\pi}{4}$ and $B = x$ into this gives

$$\cos\left(x+\frac{\pi}{4}\right)\cos(x)+\sin\left(x+\frac{\pi}{4}\right)\sin(x)=\cos\left(x+\frac{\pi}{4}-x\right)=\cos\left(\frac{\pi}{4}\right)$$

Substituting this into the above (†) we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin\left(x+\frac{\pi}{4}\right)}{1-\cos(x)}\right]=\frac{\cos\left(x+\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)}{\left[1-\cos(x)\right]^2}$$

The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=\frac{\pi}{4}$ is found by substituting $x=\frac{\pi}{4}$ into this $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)}{\left[1-\cos\left(\frac{\pi}{4}\right)\right]^2}=\frac{\cos\left(\frac{\pi}{2}\right)-\frac{1}{\sqrt{2}}}{\left(1-\frac{1}{\sqrt{2}}\right)^2}\qquad\left[\text{Because }\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\right]$$

$$=\frac{0-\frac{1}{\sqrt{2}}}{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2}=-\frac{1}{\sqrt{2}\,\frac{\left(\sqrt{2}-1\right)^2}{\left(\sqrt{2}\right)^2}}=-\frac{\sqrt{2}}{2-2\sqrt{2}+1}=-\frac{\sqrt{2}}{3-2\sqrt{2}}$$

The value is $-\frac{\sqrt{2}}{3-2\sqrt{2}}$.

(ii) *How do we differentiate* $y=\frac{\left(x^2+1\right)^2(x+7)^3}{(2x-3)}$*?*

Since this is complex function with product, quotient and indices therefore it is easier to use logarithmic differentiation. Remember logs convert multiplication and division into addition and subtraction problems which are generally easier to deal with. Taking logs of both sides of the given function and applying the basic rules of logs which are

(5.12) $\ln(AB)=\ln(A)+\ln(B)$

(5.13) $\ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)$

(5.14) $\ln\left(A^n\right)=n\ln(A)$

gives

$$\ln(y)=\ln\left[\frac{\left(x^2+1\right)^2(x+7)^3}{(2x-3)}\right]$$

$$=\ln\left[\left(x^2+1\right)^2(x+7)^3\right]-\ln(2x-3)\qquad\left[\text{Applying }\ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)\right]$$

$$=\ln\left[\left(x^2+1\right)^2\right]+\ln\left[(x+7)^3\right]-\ln(2x-3)\qquad\left[\text{By }\ln(AB)=\ln(A)+\ln(B)\right]$$

$$=2\ln\left(x^2+1\right)+3\ln(x+7)-\ln(2x-3)\qquad\left[\text{Using }\ln\left(A^n\right)=n\ln(A)\right]$$

We have not differentiated the given function but rewritten the given function in terms of logarithms. Differentiating this $\ln(y) = 2\ln(x^2+1) + 3\ln(x+7) - \ln(2x-3)$ by using $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(u)] = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(y)] = 2\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+1)\right] + 3\frac{\mathrm{d}}{\mathrm{d}x}[\ln(x+7)] - \frac{\mathrm{d}}{\mathrm{d}x}[\ln(2x-3)]$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x^2+1}(2x) + \frac{3}{x+7} - \frac{2}{2x-3} \qquad [\text{Differentiating}]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left[\frac{4x}{x^2+1} + \frac{3}{x+7} - \frac{2}{2x-3}\right] \qquad [\text{Multiplying through by } y]$$

$$= \frac{(x^2+1)^2(x+7)^3}{2x-3}\left[\frac{4x}{x^2+1} + \frac{3}{x+7} - \frac{2}{2x-3}\right] \qquad \begin{bmatrix}\text{Substituting the} \\ \text{given function } y\end{bmatrix}$$

The result is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+1)^2(x+7)^3}{2x-3}\left[\frac{4x}{x^2+1} + \frac{3}{x+7} - \frac{2}{2x-3}\right]$.

(iii) *How do we differentiate* $x^2 + y^2 + \sin(xy) = 0$ *?*

Need to use implicit differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + y^2 + \sin(xy)\right] = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) + \frac{\mathrm{d}}{\mathrm{d}x}[\sin(xy)] = 0$$

$$= 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + \cos(xy)\frac{\mathrm{d}}{\mathrm{d}x}(xy) = 0 \qquad (\dagger)$$

How do we differentiate $\frac{\mathrm{d}}{\mathrm{d}x}(xy)$ *in the last line above?*

Use the product rule because $xy = x \times y$ is a product:

$$\begin{array}{ll} u = x & v = y \\ u' = 1 & v' = \frac{\mathrm{d}y}{\mathrm{d}x} \end{array}$$

Substituting this into the product rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = u'v + uv'$$

$$= (1)y + x\frac{\mathrm{d}y}{\mathrm{d}x} = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$

Substituting this into (†) gives

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + \cos(xy)\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$$

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos(xy) + x\cos(xy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$[2x + y\cos(xy)] + [2y + x\cos(xy)]\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y\cos(xy)}{2y + x\cos(xy)}$$

We have $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y\cos(xy)}{2y + x\cos(xy)}$.

6. We need to differentiate $f(x)=\sin(\ln(2x))$. *How?*

You should know the derivative of sin is cos. Use the chain rule directly, that is

(6.19) $$\frac{d}{dx}\left[\sin(u)\right]=\cos(u)\frac{du}{dx}$$

Using this with $u=\ln(2x)$ gives the derivative of $f(x)=\sin(\ln(2x))$ as

$$f'(x)=\cos(u)\frac{du}{dx}=\cos(\ln(2x))\frac{d}{dx}(\ln(2x))$$

$$=\cos(\ln(2x))\frac{1}{\cancel{2}x}(\cancel{2})\underbrace{=}_{\text{Cancelling 2's}}\frac{\cos(\ln(2x))}{x}$$

7. *How do we differentiate* $f(x)=\ln\left(1/\left(e^{3x}+1\right)\right)$?

You should know that the derivative of $\ln(x)$ is $1/x$. We use

(6.18) $$\frac{d}{dx}\left[\ln(u)\right]=\frac{1}{u}\frac{du}{dx}$$

with $u=\frac{1}{e^{3x}+1}=\left(e^{3x}+1\right)^{-1}$. We need to differentiate this function u so that we can find $\frac{du}{dx}$:

$$\frac{du}{dx}=\frac{d}{dx}\left(e^{3x}+1\right)^{-1}$$

$$=-\left(e^{3x}+1\right)^{-2}\left(3e^{3x}\right)=-\frac{3e^{3x}}{\left(e^{3x}+1\right)^{2}}\qquad\left[\text{Remember } a^{-n}=\frac{1}{a^{n}}\right]$$

The derivative of $f(x)=\ln\left(1/\left(e^{3x}+1\right)\right)$ is

$$\frac{d}{dx}\left[\ln\left(\frac{1}{e^{3x}+1}\right)\right]=\frac{1}{u}\frac{du}{dx}$$

$$=\frac{1}{\frac{1}{e^{3x}+1}}\left(-\frac{3e^{3x}}{\left(e^{3x}+1\right)^{2}}\right)=-\frac{3e^{3x}}{e^{3x}+1}\qquad\left[\text{Cancelling }\left(e^{3x}+1\right)\right]$$

8. We need to differentiate $y=a\cosh(x/a)+c$. *How?*

Use the chain rule directly:

$$\frac{dy}{dx}=\frac{\cancel{a}}{\cancel{a}}\sinh\left(\frac{x}{a}\right)\underbrace{=}_{\text{Cancelling } a\text{'s}}\sinh\left(\frac{x}{a}\right)\qquad\left[\text{Because }\frac{d}{dx}\left[\cosh(u)\right]=\sinh(u)\frac{du}{dx}\right]$$

Need to differentiate this again:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\sinh\left(\frac{x}{a}\right)\right]=\frac{1}{a}\cosh\left(\frac{x}{a}\right)\qquad\left[\text{Because }\frac{d}{dx}\left[\sinh(u)\right]=\cosh(u)\frac{du}{dx}\right]$$

We have $\frac{d^2y}{dx^2}=\frac{1}{a}\cosh\left(\frac{x}{a}\right)$. *What do we need to show?*

The RHS of the given equation, that is $\frac{1}{a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ is equal to $\frac{1}{a}\cosh\left(\frac{x}{a}\right)$. Substituting the above result $\frac{\mathrm{d}y}{\mathrm{d}x}=\sinh\left(\frac{x}{a}\right)$ into this $\frac{1}{a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ gives

$$\frac{1}{a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\frac{1}{a}\sqrt{1+\left[\sinh\left(\frac{x}{a}\right)\right]^2}=\frac{1}{a}\sqrt{1+\sinh^2\left(\frac{x}{a}\right)}$$

How do we simplify $1+\sinh^2\left(\frac{x}{a}\right)$*?*

By using the fundamental hyperbolic identity

(5.32) $\cosh^2(t)-\sinh^2(t)=1$

Re-arranging this gives $\cosh^2(t)=1+\sinh^2(t)$. Therefore $1+\sinh^2\left(\frac{x}{a}\right)=\cosh^2\left(\frac{x}{a}\right)$.

Substituting this into the above we have

$$\frac{1}{a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\frac{1}{a}\sqrt{1+\sinh^2\left(\frac{x}{a}\right)}=\frac{1}{a}\sqrt{\cosh^2\left(\frac{x}{a}\right)}=\frac{1}{a}\cosh\left(\frac{x}{a}\right)$$

Hence we have our result because the LHS is $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{a}\cosh\left(\frac{x}{a}\right)$.

9. (a) We need to differentiate $f(x)=e^{\sin(x)}$. You should know the derivative of the exponential function:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^u\right)=e^u\frac{\mathrm{d}u}{\mathrm{d}x}$$

To find the derivative of $f(x)=e^{\sin x}$ we use this with $u=\sin(x)$:

$$\frac{\mathrm{d}f}{\mathrm{d}x}=e^{\sin(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x)\right]=e^{\sin(x)}\cos(x)$$

We need to differentiate $g(x)=x^3-\cos(2x)$:

$$\frac{\mathrm{d}g}{\mathrm{d}x}=3x^2-\left(-2\sin(2x)\right)\qquad\left[\text{Because }\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(kx)\right]=-k\sin(kx)\right]$$
$$=3x^2+2\sin(2x)$$

How do we differentiate $\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x).g(x)\right)$*?*

Use the product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x).g(x)\right)=\frac{\mathrm{d}f}{\mathrm{d}x}g(x)+\frac{\mathrm{d}g}{\mathrm{d}x}f(x)\qquad(*)$$

Substituting the above established results: $f(x)=e^{\sin(x)}$, $\frac{\mathrm{d}f}{\mathrm{d}x}=e^{\sin(x)}\cos(x)$, $g(x)=x^3-\cos(2x)$ and $\frac{\mathrm{d}g}{\mathrm{d}x}=3x^2+2\sin(2x)$ into (*) gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x).g(x)\big)=\frac{\mathrm{d}f}{\mathrm{d}x}g(x)+\frac{\mathrm{d}g}{\mathrm{d}x}f(x)$$
$$=e^{\sin(x)}\cos(x)\left[x^3-\cos(2x)\right]+\left[3x^2+2\sin(2x)\right]e^{\sin(x)}$$
$$=e^{\sin(x)}\left[x^3\cos(x)-\cos(x)\cos(2x)+3x^2+2\underbrace{\sin(2x)}_{2\sin(x)\cos(x)}\right]$$
$$=e^{\sin(x)}\left[x^3\cos(x)-\cos(x)\cos(2x)+3x^2+4\sin(x)\cos(x)\right]$$
$$=e^{\sin(x)}\left[\left(x^3-\cos(2x)+4\sin(x)\right)\cos(x)+3x^2\right]$$

(b) Given $x=2t-\cos(t)$, $y=\sin(t)$, we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by using parametric differentiation:

$$x=2t-\cos(t) \qquad\qquad y=\sin(t)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t}=2-\left[-\sin(t)\right]=2+\sin(t) \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t}=\cos(t)$$

Recall that

(6.36) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{dy/dt}{dx/dt}$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{dy/dt}{dx/dt}=\frac{\cos(t)}{2+\sin(t)}$$

Substituting $t=\frac{\pi}{2}$ into $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos(\pi/2)}{2+\sin(\pi/2)}=\frac{0}{2+1}=0$.

How do we find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ *for the given parametric equations?*

By

(6.37) $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

We already have the denominator from above $\frac{\mathrm{d}x}{\mathrm{d}t}=2+\sin(t)$. We need to find the numerator in (6.37). We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\cos(t)}{2+\sin(t)}\right] \qquad \left[\text{From above } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos(t)}{2+\sin(t)}\right]$$

How do we find the derivative of this function?

Because we have quotient function therefore we use the quotient rule:

(6.32) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}$$

Let $u=\cos(t)$ and $v=2+\sin(t)$. Differentiating these functions:

$$u'=-\sin(t) \qquad v'=\cos(t)$$

Substituting these into (6.32) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\cos(t)}{2+\sin(t)}\right]=\frac{u'v-uv'}{v^2}$$

$$=\frac{-\sin(t)\left[2+\sin(t)\right]-\cos(t)\cos(t)}{\left[2+\sin(t)\right]^2}$$

$$=\frac{-2\sin(t)-\sin^2(t)-\cos^2(t)}{\cos^2(t)}$$

$$=\frac{-2\sin(t)-\left[\sin^2(t)+\cos^2(t)\right]}{\left[2+\sin(t)\right]^2}=\frac{-2\sin(t)-1}{\left[2+\sin(t)\right]^2}\quad\begin{bmatrix}\text{Because}\\ \sin^2(t)+\cos^2(t)=1\end{bmatrix}$$

Substituting $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{-2\sin(t)-1}{\left[2+\sin(t)\right]^2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=2+\sin(t)$ into the above formula (6.37):

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\frac{-2\sin(t)-1}{\left[2+\sin(t)\right]^2}}{2+\sin(t)}=-\frac{\left(2\sin(t)+1\right)}{\left[2+\sin(t)\right]^3}$$

Substituting $t=\frac{\pi}{2}$ into this gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{\left(2\sin\left(\frac{\pi}{2}\right)+1\right)}{\left(2+\sin\left(\frac{\pi}{2}\right)\right)^3}=-\frac{2+1}{(2+1)^3}=-\frac{\cancel{3}}{3^{\cancel{3}2}}=-\frac{1}{3^2}=-\frac{1}{9}$$

Putting $t=\frac{\pi}{2}$ into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ gives $-\frac{1}{9}$.

10. (i) *How do we differentiate* $xe^{x^2}-yx^{\frac{3}{2}}=y^2\cos x-5$?

Use implicit differentiation. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{x^2}-yx^{\frac{3}{2}}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[y^2\cos x-5\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{x^2}\right]-\frac{\mathrm{d}}{\mathrm{d}x}\left[yx^{\frac{3}{2}}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[y^2\cos x\right]-\frac{\mathrm{d}}{\mathrm{d}x}(5)$$

Note that $\left[xe^{x^2}\right]$, $\left[yx^{\frac{3}{2}}\right]$ and $\left[y^2\cos x\right]$ are **ALL** products so we need to use the product rule on these.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{x^2}\right]=(1)e^{x^2}+x\left(2xe^{x^2}\right)=e^{x^2}+2x^2e^{x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[yx^{\frac{3}{2}}\right]=\frac{\mathrm{d}y}{\mathrm{d}x}x^{\frac{3}{2}}+\frac{3}{2}yx^{\frac{1}{2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y^2\cos x\right]=2y\frac{\mathrm{d}y}{\mathrm{d}x}\cos(x)-y^2\sin(x)$$

Substituting these into the above

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{x^2}\right]-\frac{\mathrm{d}}{\mathrm{d}x}\left[yx^{\frac{3}{2}}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[y^2\cos x\right]-\frac{\mathrm{d}}{\mathrm{d}x}(5)$$

gives

$$e^{x^2}+2x^2e^{x^2}-\left(\frac{\mathrm{d}y}{\mathrm{d}x}x^{\frac{3}{2}}+\frac{3}{2}yx^{\frac{1}{2}}\right)=2y\frac{\mathrm{d}y}{\mathrm{d}x}\cos(x)-y^2\sin(x)-0$$

Opening up the brackets and rearranging:

$$e^{x^2}+2x^2e^{x^2}-x^{\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{3}{2}yx^{\frac{1}{2}}=2y\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x}-y^2\sin(x)$$

$$e^{x^2}+2x^2e^{x^2}+y^2\sin(x)-\frac{3}{2}yx^{\frac{1}{2}}=2y\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x}+x^{\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$2e^{x^2}+4x^2e^{x^2}+2y^2\sin(x)-3yx^{\frac{1}{2}}=\left[4y\cos(x)+2x^{\frac{3}{2}}\right]\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\left[\begin{array}{l}\text{Multiplying}\\\text{through by 2}\end{array}\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2e^{x^2}+4x^2e^{x^2}+2y^2\sin(x)-3yx^{\frac{1}{2}}}{4y\cos(x)+2x^{\frac{3}{2}}}\qquad\left[\text{Making }\frac{\mathrm{d}y}{\mathrm{d}x}\text{ the subject}\right]$$

The result of differentiating the given function is $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2e^{x^2}+4x^2e^{x^2}+2y^2\sin(x)-3yx^{\frac{1}{2}}}{4y\cos(x)+2x^{\frac{3}{2}}}$.

(ii) To find the coordinates of the point P we substitute $t=1$ into

$$x=t-\sin\left(\frac{\pi t}{2}\right),\qquad y=\cos\left(\frac{\pi t}{2}\right)-t^2$$

$$x=1-\sin\left(\frac{\pi}{2}\right)=1-1=0\qquad y=\cos\left(\frac{\pi}{2}\right)-1^2=0-1=-1$$

Thus the point P is given by $(0,\ -1)$. *How do we find* $\frac{\mathrm{d}y}{\mathrm{d}x}$ *in this case?*

Use parametric differentiation. Remember $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(kx)\right]=k\cos(kx)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(kx)\right]=-k\sin(kx)$:

$$x=t-\sin\left(\frac{\pi t}{2}\right),\qquad y=\cos\left(\frac{\pi t}{2}\right)-t^2$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=1-\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)\qquad\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)-2t$$

Recall $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$. Substituting the above into this yields:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{-\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)-2t}{1-\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)}$$

Substituting $t=1$ into this gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)-2}{1-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)}=\frac{-\frac{\pi}{2}-2}{1-\frac{\pi}{2}(0)}=-\frac{\pi}{2}-2$$

At $t=1$ we have the derivative equal to $-\frac{\pi}{2}-2$.

11. *How do we find* $\frac{\mathrm{d}y}{\mathrm{d}x}$ *given* $\ln(e+y)=e^{\sin(x+y)}$?

Use implicit differentiation. You should know the derivatives of the ln and exponential functions because these are common functions which crop up in many areas of engineering and science. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(u)\right)=\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\qquad\qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right)=e^{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Applying these to the given implicit function we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(e+y)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin(x+y)}\right]$$

$$\frac{1}{e+y}\frac{\mathrm{d}y}{\mathrm{d}x}=e^{\sin(x+y)}\cos(x+y)\left[1+\frac{\mathrm{d}y}{\mathrm{d}x}\right]$$

$$=e^{\sin(x+y)}\cos(x+y)+\frac{\mathrm{d}y}{\mathrm{d}x}\left[e^{\sin(x+y)}\cos(x+y)\right]$$

Putting $x=0$ and $y=0$ into this $\frac{1}{e+y}\frac{\mathrm{d}y}{\mathrm{d}x}=e^{\sin(x+y)}\cos(x+y)+\frac{\mathrm{d}y}{\mathrm{d}x}\left[e^{\sin(x+y)}\cos(x+y)\right]$ gives:

$$\frac{1}{e+0}\frac{\mathrm{d}y}{\mathrm{d}x}=e^{\sin(0+0)}\cos(0+0)+\frac{\mathrm{d}y}{\mathrm{d}x}\left[e^{\sin(0+0)}\cos(0+0)\right]$$

$$\frac{1}{e}\frac{\mathrm{d}y}{\mathrm{d}x}=e^{0}\cos(0)+\frac{\mathrm{d}y}{\mathrm{d}x}\left[e^{0}\cos(0)\right]=1+\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\left[\text{Because } e^{0}=1,\ \cos(0)=1\right]$$

Collecting the $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms onto one side of this $\frac{1}{e}\frac{\mathrm{d}y}{\mathrm{d}x}=1+\frac{\mathrm{d}y}{\mathrm{d}x}$ yields

$$\frac{1}{e}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

$$\left(\frac{1}{e}-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1\qquad\text{implies that}\qquad\left(\frac{1-e}{e}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

Transposing to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject we have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{e}{1-e}$$

12. (a) *How do we find* $f'(x)$ *given* $f(x)=3^{x\ln(x)}$?

We use

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{u}\right)=a^{u}\ln(a)\frac{\mathrm{d}u}{\mathrm{d}x}\tag{6.17}$$

with $u=x\ln(x)$ and $a=3$. *How do we differentiate this* $u=x\ln(x)$?

Apply the product rule

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1)\ln(x) + x\frac{1}{x} = \ln(x) + 1$$

We have

$$f'(x) = a^u \ln(a)\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{where } a = 3 \text{ and } u = x\ln(x)$$

$$= 3^{x\ln(x)}\ln(3)\left[\ln(x)+1\right]$$

Hence $f'(x) = 3^{x\ln(x)}\ln(3)\left[\ln(x)+1\right]$.

(b) Need to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x\ln(x^4)}\right)$. First we rewrite the square root $\sqrt{\ }$ as power ½ and then we use the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x\ln(x^4)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x\ln(x^4)\right]^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left[x\ln(x^4)\right]^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\underbrace{\left[x\ln(x^4)\right]}_{\substack{\text{Apply the product rule}\\ \text{with } u=x \text{ and } v=\ln(x^4)}}$$

$$= \frac{1}{2}\left[x\ln(x^4)\right]^{-\frac{1}{2}}\left[(1)\ln(x^4) + x\frac{1}{x^4}(4x^3)\right]$$

$$= \frac{\ln(x^4)+4}{2\sqrt{x\ln(x^4)}} \qquad \left[\text{Applying } a^{-1/2} = \frac{1}{\sqrt{a}}\right]$$

(c) Need to determine $\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\ln(x)\right)^{\cos(x)}\right)$. *How?*

Use logarithmic differentiation. Let $y = \left(\ln(x)\right)^{\cos(x)}$. Taking logs of both sides:

$$\ln(y) = \ln\left[\left(\ln(x)\right)^{\cos(x)}\right] = \cos(x)\ln\left(\ln(x)\right) \qquad \left[\text{By } \ln(A^n) = n\ln(A)\right]$$

How do we differentiate $\cos(x)\ln\left(\ln(x)\right)$ *?*

Use the product rule with $u = \cos(x)$ and $v = \ln\left(\ln(x)\right)$. Differentiating these terms we have $u' = -\sin(x)$ and

$$v' = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\ln(x)\right)\right] = \frac{1}{\ln(x)}\frac{1}{x} \qquad \left[\text{Using } \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(u)\right] = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}\right]$$

Putting these into the product rule formula

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x)\ln(\ln(x))\right]=u'v+uv'$$

$$=-\sin(x)\ln(\ln(x))+\cos(x)\frac{1}{\ln(x)}\frac{1}{x}$$

$$=\frac{-x\ln(x)\sin(x)\ln(\ln(x))+\cos(x)}{x\ln(x)} \qquad \left[\text{Common denominator}\right]$$

$$=\frac{\cos(x)-x\ln(x)\sin(x)\ln(\ln(x))}{x\ln(x)}$$

We have differentiated the RHS of $\ln(y)=\ln\left[(\ln(x))^{\cos(x)}\right]$. Differentiating the LHS we have $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(y)\right]=\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$. Equating these gives

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos(x)-x\ln(x)\sin(x)\ln(\ln(x))}{x\ln(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left[\frac{\cos(x)-x\ln(x)\sin(x)\ln(\ln(x))}{x\ln(x)}\right]$$

$$=(\ln(x))^{\cos(x)}\left[\frac{\cos(x)-x\ln(x)\sin(x)\ln(\ln(x))}{x\ln(x)}\right] \qquad \left[\begin{array}{l}\text{Substituting the given function}\\ y=(\ln(x))^{\cos(x)}\end{array}\right]$$

$$=(\ln(x))^{\cos(x)-1}\left[x^{-1}\cos(x)-\ln(x)\sin(x)\ln(\ln(x))\right] \qquad \left[\begin{array}{l}\text{Using the rules of}\\ \text{indices on base }\ln(x)\end{array}\right]$$

The answer is $(\ln(x))^{\cos(x)-1}\left[x^{-1}\cos(x)-\ln(x)\sin(x)\ln(\ln(x))\right]$.

(d) *How do we find* $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}\right)$*?*

Because the given function consists of products, quotients and indices it is therefore easier to use logarithmic differentiation. Let $y=\frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}$. Taking logs of both sides:

$$\ln(y)=\ln\left[\frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}\right]$$

$$=\ln(4x-1)^3-\ln\left[(2x^2-1)^{3/2}(x+1)^2\right] \qquad \left[\text{Applying }\ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)\right]$$

$$=\ln(4x-1)^3-\left[\ln(2x^2-1)^{3/2}+\ln(x+1)^2\right] \qquad \left[\text{By }\ln(AB)=\ln(A)+\ln(B)\right]$$

$$=3\ln(4x-1)-\frac{3}{2}\ln(2x^2-1)-2\ln(x+1) \qquad \left[\text{By }\ln(A^n)=n\ln(A)\right]$$

It is easier to differentiate this because logs has reduced the given function to a problem of sums and differences rather than products, quotients and indices.

We need to differentiate $\ln(y) = 3\ln(4x-1) - \frac{3}{2}\ln(2x^2-1) - 2\ln(x+1)$. We can separate these as follows:

$$\frac{d}{dx}[\ln(y)] = 3\frac{d}{dx}[\ln(4x-1)] - \frac{3}{2}\frac{d}{dx}[\ln(2x^2-1)] - 2\frac{d}{dx}[\ln(x+1)]$$

Differentiating each part gives

$$\frac{1}{y}\frac{dy}{dx} = 3\frac{1}{4x-1}(4) - \frac{3}{2}\frac{1}{2x^2-1}(4x) - 2\frac{1}{x+1}(1)$$

$$= \frac{12}{4x-1} - \frac{6x}{2x^2-1} - \frac{2}{x+1}$$

$$\frac{dy}{dx} = y\left[\frac{12}{4x-1} - \frac{6x}{2x^2-1} - \frac{2}{x+1}\right]$$

$$= \frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}\left[\frac{12}{4x-1} - \frac{6x}{2x^2-1} - \frac{2}{x+1}\right] \qquad \left[\text{Substituting the given function } y = \frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}\right]$$

Hence $\frac{d}{dx}\left(\frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}\right)$ is equal to $\frac{(4x-1)^3}{(2x^2-1)^{3/2}(x+1)^2}\left[\frac{12}{4x-1} - \frac{6x}{2x^2-1} - \frac{2}{x+1}\right]$.

(e) *How do we find* $\frac{d}{dx}\left(x^2\cos(x)e^{3x}\sin(\pi/2)\right)$ *?*

First note that $\sin\left(\frac{\pi}{2}\right) = 1$. Let y be the given function and substituting this yields

$$y = x^2\cos(x)e^{3x}\sin(\pi/2) = x^2\cos(x)e^{3x}$$

Again it is easier to take logs of both sides:

$$\ln(y) = \ln\left[x^2\cos(x)e^{3x}\right]$$

$$= \ln(x^2) + \ln(\cos(x)) + \ln(e^{3x})$$

$$= 2\ln(x) + \ln(\cos(x)) + 3x$$

Differentiating each part gives

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x} + \frac{1}{\cos(x)}\sin(x) + 3$$

$$= \frac{2}{x} + \tan(x) + 3 \qquad \left[\text{Remember } \frac{\sin}{\cos} = \tan\right]$$

Multiplying through by y:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(\frac{2}{x} + \tan(x) + 3\right)$$

$$= x^2\cos(x)e^{3x}\left(\frac{2}{x} + \tan(x) + 3\right) \qquad \left[\text{Substituting the given function } y = x^2\cos(x)e^{3x}\right]$$

$$= x^{\cancel{2}}\cos(x)e^{3x}\left(\frac{2 + x\tan(x) + 3x}{\cancel{x}}\right) \qquad \left[\text{Writing the bracket term with common denominator}\right]$$

$$= x\cos(x)e^{3x}\left(2 + x\tan(x) + 3x\right) \qquad \left[\text{Cancelling the } x\text{'s}\right]$$