Complete Solutions to Examination Questions 7

1. We have:

(i) The velocity v is given by differentiating the distance $s = 30t - 6t^2$ with respect to time t:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 30 - 12t$$

The initial velocity is found by substituting $t = 0$ into $v = 30 - 12t$:

$$v = 30 - (12 \times 0) = 30$$

The initial velocity is 30m/s.

(ii) The velocity after 3 seconds is given by substituting $t = 3$ into $v = 30 - 12t$:

$$v = 30 - (12 \times 3) = -6$$

We have after 3 seconds the velocity is -6 m/s.

(iii) The acceleration a is determined by differentiating the velocity with respect to time:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[30 - 12t] = -12$$

The constant acceleration is $-12\ \mathrm{m/s^2}$.

2. (a) To determine the maxima and minima of $y = x^3 - 3x + 1$ we need to first find the stationary points. *How do we find the stationary points?*

The stationary points occur where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Therefore differentiating the given function we have

$$y = x^3 - 3x + 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3 = 0$$

Solving the quadratic yields:

$$3x^2 - 3 = 3\left(x^2 - 1\right) = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

To distinguish between the two points at $x = -1$ and $x = +1$ we differentiate again:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x$$

Substituting $x = -1$ into $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x$ gives $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6 \times (-1) = -6 < 0$ [Negative]. Hence we have maximum at $x = -6$.

Similarly substituting $x = +1$ into $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x$ gives $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6 \times (1) = 6 > 0$ [Positive] therefore we have a minimum at $x = +1$.

The corresponding y values are found by substituting $x = -1$ and $x = +1$ into $y = x^3 - 3x + 1$:

$$x = -1, \quad y = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$$

$$x = +1, \quad y = 1^3 - 3(1) + 1 = 1 - 3 + 1 = -1$$

We have local maximum at $(-1, \quad 3)$ and local minimum at $(1, \quad -1)$.

(b) (i) Substituting $t=2$ into $x=t^3-2t^2+t$ gives

$$x=2^3-2(2)^2+2=2$$

The particle is 2m away from **Z**.

(ii) The speed *v* of the particle is given by differentiating *x* with respect to *t*:

$$x=t^3-2t^2+t$$

$$v=\frac{dx}{dt}=3t^2-4t+1$$

The speed after 2 seconds is $v=3(2)^2-4(2)+1=5\,\text{m/s}$.

(iii) The acceleration *a* after 2 seconds is determined by differentiating *v*:

$$a=\frac{dv}{dt}=\frac{d}{dt}\left[3t^2-4t+1\right]$$

$$=6t-4$$

Substituting $t=2$ into $a=6t-4$ gives the acceleration after 2 seconds, $a=6(2)-4=8\,\text{m/s}^2$.

(iv) The particle is at rest when $x=0$. Equating our given *x* to zero and solving we have

$$t^3-2t^2+t=0$$

$$t\left(t^2-2t+1\right)=0$$

$$t(t-1)^2=0 \quad \text{gives} \quad t=0, \quad t=1$$

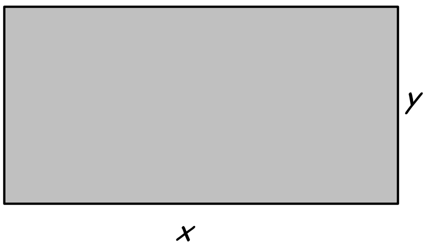
The particle is at rest when $t=0$ or $t=1$s.

(v) The acceleration $a=6t-4$ is zero at

$$6t-4=0 \quad \text{gives} \quad t=\frac{4}{6}=\frac{2}{3}$$

The acceleration is zero when $t=\frac{2}{3}$s.

3. Let the rectangle have dimensions *x* and *y*, that is we have



The perimeter *P* is 20 so we have

$$2x+2y=20$$

$$x+y=10 \qquad \text{[Dividing through by 2]}$$

$$y=10-x$$

The area *A* of the rectangle is

$$A=xy=x(10-x)=10x-x^2 \qquad \text{[Substituting } y=10-x\text{]}$$

Differentiating this and equating to zero gives a stationary point of *A*:

$$A=10x-x^2$$

$$\frac{dA}{dx}=10-2x=0 \quad \text{yields} \quad x=5$$

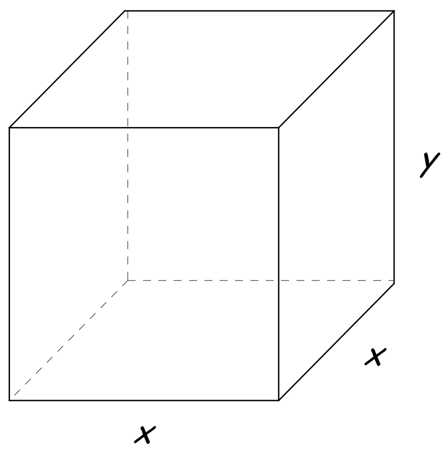
We have a stationary point of A at $x = 5$. *How do we find whether this gives a max or min?* Differentiate again:

$$\frac{dA}{dx} = 10 - 2x$$

$$\frac{d^2A}{dx^2} = -2 < 0$$

Hence $x = 5$ gives maximum area A. *What is y equal to?*
Substituting $x = 5$ into $y = 10 - x = 10 - 5 = 5$. Hence we have maximum area at $x = y = 5$ feet (a square).

4. We have the following box:



We are given that volume of the box is 12 therefore we have

$$x^2 y = 12 \quad \Rightarrow \quad y = \frac{12}{x^2} \qquad (\dagger)$$

To get the least expensive box we need to use the least amount of material or otherwise minimise the total surface area A. *What is the total surface area A equal to?*

$$A = x^2 + 4xy \quad \text{[base plus the four sides]}$$

By (†) we substitute $y = \frac{12}{x^2}$ into A:

$$A = x^2 + 4xy = x^2 + 4x\left(\frac{12}{x^2}\right)$$
$$= x^2 + 48x^{-1}$$

The cost C of the material is given by $C = 30x^2 + \left(10 \times 48x^{-1}\right) = 30x^2 + 480x^{-1}$.

We need to differentiate this in order to find the minimum cost:

$$C = 30x^2 + 480x^{-1}$$

$$\frac{dC}{dx} = 60x - 480x^{-2} = 60x - \frac{480}{x^2} = 0$$

Solving this equation $60x - \frac{480}{x^2} = 0$:

$$60x = \frac{480}{x^2} \quad \Rightarrow \quad 60x^3 = 480 \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2$$

In order to confirm that this indeed does give the minimum surface area we have to differentiate again:

$$\frac{\mathrm{d}C}{\mathrm{d}x} = 60x - 480x^{-2}$$
$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = 60 + 960x^{-3}$$

Substituting $x = 2$ into the second derivative yields $\frac{\mathrm{d}^2C}{\mathrm{d}x^2} > 0$ [Positive] which means when $x = 2$ we have minimum cost. *What is the value of y?*
Substituting $x = 2$ into $y = \frac{12}{x^2}$ we have $y = \frac{12}{2^2} = 3$. The costs are minimised when $x = 2$ in. and $y = 3$ in..

5. The Maclaurin series for $\frac{e^{2x}}{1+x} = e^{2x}(1+x)^{-1}$ can be found by using the series for e^x:

(7.15) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

We need to replace x with $2x$ in this series

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots \quad \text{[Simplifying the coefficients]}$$

We also need to find the binomial expansion of $(1+x)^{-1}$:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

Multiplying these two together gives

$$\frac{e^{2x}}{1+x} = e^{2x}(1+x)^{-1}$$
$$= \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots\right)\left(1 - x + x^2 - x^3 + x^4 - \dots\right)$$
$$= 1 - x + x^2 - x^3 + 2x\left(1 - x + x^2\right) + 2x^2(1-x) + \frac{4}{3}x^3 + \dots$$
$$= 1 - x + x^2 - x^3 + 2x - 2x^2 + 2x^3 + 2x^2 - 2x^3 + \frac{4}{3}x^3 + \dots$$
$$= 1 + x + x^2 + \frac{1}{3}x^3 + \dots$$

The first four non-zero terms are $1 + x + x^2 + \frac{1}{3}x^3 + \dots$

6. (i) Applying the binomial series

(7.24) $$(1+x)^n = 1 + nx + \left[\frac{n(n-1)}{2!}\right]x^2 + \left[\frac{n(n-1)(n-2)}{3!}\right]x^3 + \dots$$

to the given binomial term $(1+3x)^{1/3}$ with $n = 1/3$ and x replaced with $3x$:

$$\left(1+3x\right)^{1/3}=1+\frac{1}{3}\left(3x\right)+\left[\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\right]\left(3x\right)^2+\left[\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\right]\left(3x\right)^3+\ldots$$

$$=1+x+\left[\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\right]9x^2+\left[\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\right]27x^3+\ldots$$

$$=1+x+\left[-\frac{1}{9}\right]9x^2+\left[\left(\frac{10}{27}\right)\frac{1}{6}\right]27x^3+\ldots$$

$$=1+x-x^2+\frac{5}{3}x^3+\ldots \qquad \text{[Cancelling 9's and 27's]}$$

(ii) We know

(7.15) $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots \quad\Rightarrow\quad e^{-x^2}=1+\left(-x^2\right)+\frac{\left(-x^2\right)^2}{2!}+\ldots$$

(7.17) $$\cos\left(x\right)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots \quad\Rightarrow\quad \cos\left(2x\right)=1-\frac{\left(2x\right)^2}{2!}+\frac{\left(2x\right)^4}{4!}-\ldots$$

Substituting these into the given product function $e^{-x^2}\cos\left(2x\right)$ we have

$$e^{-x^2}\cos\left(2x\right)=\left[1-x^2+\frac{\left(-x^2\right)^2}{2!}-\ldots\right]\left[1-\frac{\left(2x\right)^2}{2!}+\frac{\left(2x\right)^4}{4!}-\ldots\right]$$

$$=\left[1-x^2+\frac{x^4}{2}-\ldots\right]\left[1-\frac{4x^2}{2}+\frac{16x^4}{24}-\ldots\right]$$

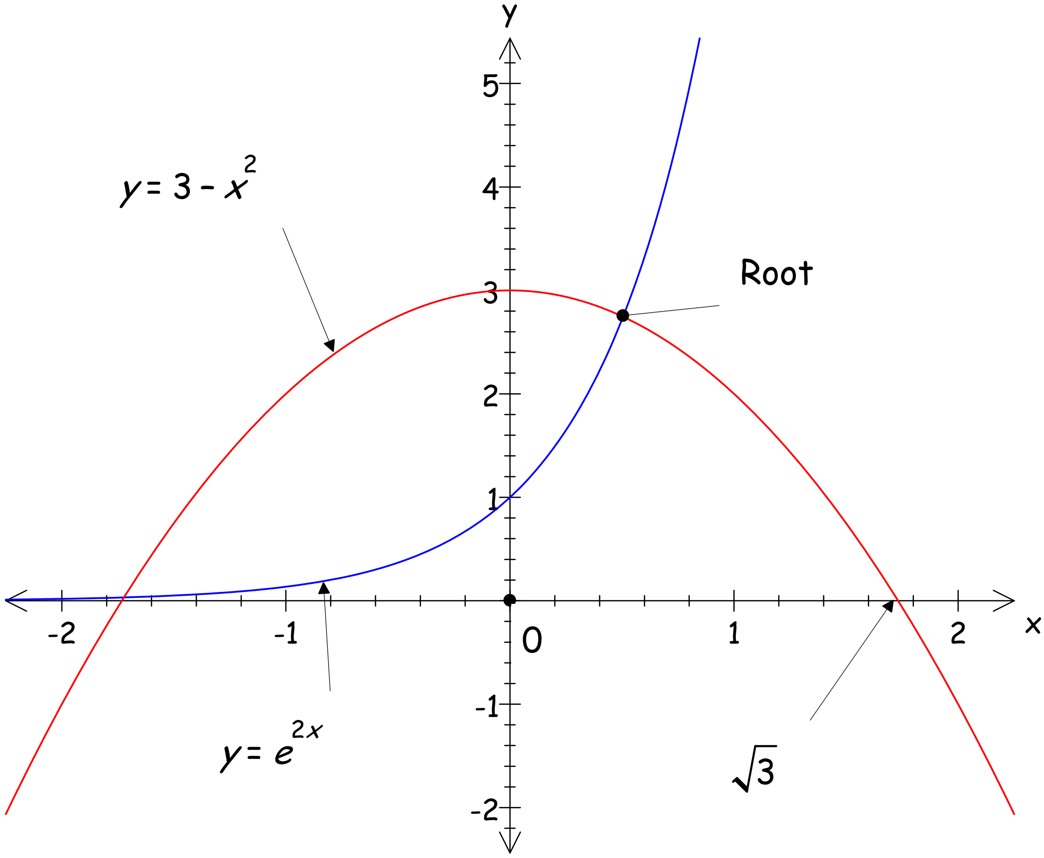
$$=1\left[1-2x^2+\frac{2}{3}x^4-\ldots\right]-x^2\left[1-2x^2+\ldots\right]+\frac{x^4}{2}\left[1-2x^2+\ldots\right]$$

$$=1-2x^2+\frac{2}{3}x^4-x^2+2x^4+\frac{x^4}{2}+\ldots \qquad \text{[Expanding]}$$

$$=1-3x^2+\left(\frac{2}{3}+\frac{1}{2}+2\right)x^4+\ldots=1-3x^2+\frac{19}{6}x^4+\ldots$$

Our expansion gives $1-3x^2+\frac{19}{6}x^4+\ldots$.

7. The graph of $y=e^{2x}$ is our normal exponential graph which crosses the *y*-axis at 1. The graph of the quadratic $y=3-x^2$ crosses the *x*-axis at $3-x^2=0$ which gives $x=\pm\sqrt{3}$. At $x=0$, $y=3$ and this is the maximum value. The two graphs are:



The root of the equation $e^{2x} = 3 - x^2$ is the intersection of the two graphs $y = e^{2x}$ and $y = 3 - x^2$ as shown above. One of the roots lies between 0 and $\sqrt{3}$ as seen above but you can show this by algebraic means as follows:

Let $f(x) = e^{2x} + x^2 - 3$ then substituting $x = 0$ and $x = \sqrt{3}$ gives

$$f(0) = e^0 + 0^2 - 3 = 1 - 3 = -2 \qquad \text{[Negative]}$$

$$f\left(\sqrt{3}\right) = e^{2\sqrt{3}} + \left(\sqrt{3}\right)^2 - 3 = e^{2\sqrt{3}} + 3 - 3 = e^{2\sqrt{3}} > 0 \qquad \text{[Positive]}$$

Since the graph goes from negative to positive between 0 and $\sqrt{3}$ therefore there must be a root between 0 and $\sqrt{3}$.

The Newton Raphson formula is

(7.25) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

With $f(x) = e^{2x} + x^2 - 3$ we have $f(x_n) = e^{2x_n} + x_n^{\,2} - 3$ and

$$f'(x_n) = 2e^{2x_n} + 2x_n \qquad \text{[Differentiating]}$$

Substituting these into (7.25) gives our result:

$$x_{n+1} = x_n - \left(\frac{e^{2x_n} + x_n^{\,2} - 3}{2e^{2x_n} + 2x_n}\right) \qquad (*)$$

By looking at the above graphs we consider a first approximation $x_1 = 0.5$. Putting $n = 1$ and $x_1 = 0.5$ into (*) yields:

$$x_2 = x_1 - \left(\frac{e^{2x_1} + x_1{}^2 - 3}{2e^{2x_1} + 2x_1}\right) = 0.5 - \left(\frac{e^{2(0.5)} + 0.5^2 - 3}{2e^{2(0.5)} + 2(0.5)}\right)$$

$$= 0.5 - \left(\frac{-0.03172}{6.43656}\right) = 0.50493$$

Substituting $x_2 = 0.50493$ and $n = 2$ into (*) gives

$$x_3 = x_2 - \left(\frac{e^{2x_2} + x_2{}^2 - 3}{2e^{2x_2} + 2x_2}\right) = 0.50493 - \left(\frac{e^{(2\times 0.50493)} + (0.50493)^2 - 3}{2e^{(2\times 0.50493)} + 2(0.50493)}\right)$$

$$= 0.50493 - \left(\frac{0.00017}{6.50029}\right) = 0.50490$$

Note that x_2 rounded to 4dp is 0.5049 and x_3 rounded to 4dp is also 0.5049. Since these agree to 4dp therefore our approximation to the root to 4dp is 0.5049.

8. (a) The equation of the tangent to $y = x^3 - 6x$ is given by differentiating this *y* with respect to *x* to give the gradient.

$$y = x^3 - 6x$$

$$\frac{dy}{dx} = 3x^2 - 6$$

The gradient *m* at $(2, \ -4)$ is found by substituting $x = 2$ into this $\frac{dy}{dx} = 3x^2 - 6$:

$$m = \frac{dy}{dx} = 3(2)^2 - 6 = 6$$

The equation of the tangent is given by $y = mx + c$ where $m = 6$. We also know that the line goes through the point $(2, \ -4)$ which means that when $x = 2$, $y = -4$. Substituting these into $y = mx + c = 6x + c$ gives

$$-4 = 6(2) + c \quad \Rightarrow \quad c = -16$$

The equation of the tangent is $y = 6x - 16$.

(b) The maximum of $f(x) = 2x^2 - x^4$ can be determined by differentiating this function and equating the result to zero which gives the stationary points:

$$f(x) = 2x^2 - x^4$$

$$f'(x) = 4x - 4x^3 = 0$$

How do we solve $4x - 4x^3 = 0$ *?*

Factorise out the $4x$:

$$4x - 4x^3 = 4x(1 - x^2) = 0 \quad \text{gives} \quad x = 0 \ \text{ or } \ x = -1, \ +1$$

To find which of these stationary points gives a maximum we need to differentiate again.

$$f'(x) = 4x - 4x^3$$

$$f''(x) = 4 - 12x^2$$

Substituting $x = -1$ or $x = +1$ gives

$$f''(x) = 4 - 12 = -8 < 0 \qquad \text{[Negative]}$$

Both these *x* values $x = -1$ and $x = +1$ yields a maximum and the maximum value is

$$f(1)=2(1)^2-1^4=1 \quad [\text{Because } f(x)=2x^2-x^4]$$

The maximum value is 1 and it occurs at $x=-1$ and $x=+1$.
[At $x=0$ we have a minimum].

9. *How do we find the binomial expansion of* $\left(1-x^2\right)^{-1/2}$ *?*

By applying the binomial series

(7.24) $$(1+x)^n=1+nx+\left[\frac{n(n-1)}{2!}\right]x^2+\left[\frac{n(n-1)(n-2)}{3!}\right]x^3+\ldots$$

with $n=-\dfrac{1}{2}$ and x being replaced by $-x^2$:

$$\left(1-x^2\right)^{-1/2}=1-\frac{1}{2}\left(-x^2\right)+\left[\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\right]\left(-x^2\right)^2+\left[\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\right]\left(-x^2\right)^3+\ldots$$

$$=1+\frac{1}{2}x^2+\left[\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}\right]x^4+\left[\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\right]\left(-x^6\right)+\ldots \qquad [\text{Simplifying}]$$

$$=1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{15}{48}x^6+\ldots=1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\ldots$$

The binomial series is $\left(1-x^2\right)^{-1/2}=1+\dfrac{1}{2}x^2+\dfrac{3}{8}x^4+\dfrac{5}{16}x^6+\ldots$.

How do we use this series to find $(0.99)^{-1/2}$ *?*

Substitute $x=0.1$ into $\left(1-x^2\right)^{-1/2}=1+\dfrac{1}{2}x^2+\dfrac{3}{8}x^4+\dfrac{5}{16}x^6+\ldots$ because $\left(1-0.1^2\right)^{-1/2}=(1-0.01)^{-1/2}=(0.99)^{-1/2}$.

$$(0.99)^{-1/2}=\left(1-0.1^2\right)^{-1/2}=1+\frac{1}{2}(0.1)^2+\frac{3}{8}(0.1)^4+\frac{5}{16}(0.1)^6+\ldots$$

$$=1+\frac{1}{2}(0.01)+\frac{3}{8}(0.0001)+\frac{5}{16}(0.000001)+\ldots$$

$$=1.005038$$

10. (a) *How do we write down the general term for* $\sum_{r=1}^{\infty}u_r=x-\dfrac{x^2}{4}+\dfrac{x^3}{9}-\dfrac{x^4}{16}\ldots$ *?*

Neglecting the sign in front of the term we have $\dfrac{x^r}{r^2}$. Since we have an alternating series where the terms oscillate between plus and minus therefore we can write this as $(-1)^{r+1}$.

Thus the general term is given by $u_r=\dfrac{(-1)^{r+1}x^r}{r^2}$. The series is

$$x-\frac{x^2}{4}+\frac{x^3}{9}-\frac{x^4}{16}...=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}x^r}{r^2}$$

(b) We use the ratio test to find the radius of convergence of this series. Let $u_r=\frac{(-1)^{r+1}x^r}{r^2}$

then $u_{r+1}=\frac{(-1)^{(r+1)+1}x^{r+1}}{(r+1)^2}=\frac{(-1)^{r+2}x^{r+1}}{(r+1)^2}$. Applying the ratio test (7.30) we have

$$L=\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right|=\lim_{r\to\infty}\left|\frac{(-1)^{r+2}x^{r+1}}{(r+1)^2}\div\frac{(-1)^{r+1}x^r}{r^2}\right|$$

$$=\lim_{r\to\infty}\left|\frac{(-1)^{r+1}(-1)x^r x}{(r+1)^2}\times\frac{r^2}{(-1)^{r+1}x^r}\right| \qquad \left[\begin{array}{l}\text{Inverting the second}\\ \text{fraction and multiplying}\end{array}\right]$$

$$=\lim_{r\to\infty}\left|(-1)\frac{x}{(r+1)^2}\times r^2\right| \qquad \left[\begin{array}{l}\text{Cancelling Out}\\ (-1)^{r+1}\ \text{ and }\ x^r\end{array}\right]$$

$$=\lim_{r\to\infty}\left|(-1)x\left(\frac{r}{r+1}\right)^2\right|$$

$$=\lim_{r\to\infty}|-1||x|\left(\frac{1}{1+1/r}\right)^2 \qquad \left[\begin{array}{l}\text{Dividing Numerator and}\\ \text{Denominator by } r\end{array}\right]$$

$$=|x|(1) \qquad \left[\text{Because as } r\to\infty,\ \frac{1}{1+1/r}\to\frac{1}{1+0}=1\right]$$

We have $L=|x|$. Remember the ratio test states that the series converges for $L<1$ therefore the series converges for $|x|<1$ which means that the radius of convergence R is equal to 1. The ratio test fails for $L=|x|=1$ which means $x=-1$ or $x=+1$.

Does the series converge at $x=+1$?

Substituting $x=+1$ into the given series we have

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}1^r}{r^2}=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}$$

This series converges because it is the p-series with $p=2$. Hence the series converges at $x=R$.

We do **not** have to test this series at $x=-1$ because the question only stipulates to test the convergence at $x=R$.

11. We can find the radius of convergence of $\sum 2^n n^3 x^n$ by using the ratio test (7.30).

Let $a_n=2^n n^3 x^n$ then $a_{n+1}=2^{n+1}(n+1)^3x^{n+1}$. Substituting these into $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)^3 x^{n+1}}{2^n n^3 x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^n 2(n+1)^3 x^n x}{2^n n^3 x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(n+1)^3 x}{n^3}\right| \qquad \left[\text{Cancelling } 2^n \text{ and } x^n\right]$$

$$= \lim_{n\to\infty}\left|2x\left(\frac{n+1}{n}\right)^3\right| \qquad \left[\text{Because } \frac{(n+1)^3}{n^3} = \left(\frac{n+1}{n}\right)^3\right]$$

$$= \lim_{n\to\infty}\left|2x\left(1+\frac{1}{n}\right)^3\right| = |2x|(1) \qquad \left[\text{Because as } n\to\infty,\ \ 1+\frac{1}{n}\to 1+0=1\right]$$

The series converges for $L = |2x| = 2|x| < 1$. This means the series converges for $|x| < \frac{1}{2}$ which gives the radius of convergence R equal to ½.

12. The Taylor series expansion about the point $x=0$ is the Maclaurin series. The Maclaurin series for e^x is given by

(7.15) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

To find the expansion of $\left(1+e^{x^2}\right)(1-x)$ we use (7.15) with x replaced with x^2:

$$\left(1+e^{x^2}\right)(1-x) = \left[1 + \underbrace{\left(1 + x^2 + \frac{\left(x^2\right)^2}{2!} + \dots\right)}_{=e^{x^2}}\right](1-x)$$

$$= \left[2 + x^2 + \frac{x^4}{2} + \dots\right](1-x) \qquad [\text{Simplifying}]$$

$$= (2-2x) + \left(x^2 - x^3\right) + \left(\frac{x^4}{2} - \frac{x^5}{2}\right) + \dots \qquad [\text{Expanding}]$$

$$= 2 - 2x + x^2 - \dots$$

13. The gradient of the tangent to $y = (2+x)e^{-x}$ at $(0,\ \ 2)$ is found by differentiating this:

$$y = (2+x)e^{-x}$$

$$\frac{dy}{dx} = (1)e^{-x} - (2+x)e^{-x} \qquad \left[\text{Using the product rule } (uv)' = u'v + v'u\right]$$

$$= (1-2-x)e^{-x} \qquad [\text{Factorising}]$$

$$= (-1-x)e^{-x} = -(1+x)e^{-x}$$

Subs $x=0$, because we want to find the tangent at $(0,\quad 2)$, into $m=\dfrac{\mathrm{d}y}{\mathrm{d}x}=-(1+x)e^{-x}$:

$$m=\frac{\mathrm{d}y}{\mathrm{d}x}=-(1+0)e^{-0}=-1 \qquad [\text{Gradient}]$$

The equation of the tangent is $y=mx+c=-x+c$. *How do we find the value of* c*?*

Since the tangent goes through the point $(0,\quad 2)$ we know it satisfies that when $x=0$, $y=2$. Putting these values into $y=-x+c$ gives

$$2=0+c$$

The equation of the tangent is $y=2-x$.

14. We need to find the tangent line to $x^2+xy+y^2=e^{y-2}+6$. Similar to the solution of question 13. We first find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2+xy+y^2\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{y-2}+6\right]$$

$$2x+\underbrace{\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)}_{\text{Product rule}}+2y\frac{\mathrm{d}y}{\mathrm{d}x}=e^{y-2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$2x+y=\left(e^{y-2}-x-2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \left[\text{Collecting the } \frac{\mathrm{d}y}{\mathrm{d}x} \text{ terms on one side}\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+y}{e^{y-2}-x-2y}$$

This is the gradient function of $x^2+xy+y^2=e^{y-2}+6$. *How do we find the gradient at* $(1,\quad 2)$*?*

Substitute $x=1,\ y=2$ into $m=\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x+y}{e^{y-2}-x-2y}$:

$$m=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(1)+2}{e^{2-2}-1-2(2)}=\frac{4}{-4}=-1$$

The equation of the tangent is $y=mx+c=-x+c$ and goes through the point $(1,\quad 2)$:

$$2=-1+c \quad \text{gives} \quad c=3$$

Hence the equation of the tangent is $y=3-x$.

15. To sketch the graph of $P(r)=\left(\dfrac{4r^2}{a^2}\right)e^{-2r/a}$ we need to differentiate this function with respect to r to find the stationary points.

$$P'(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{4r^2}{a^2}\right)e^{-2r/a}\right]$$

$$= \frac{4}{a^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2 e^{-2r/a}\right] \qquad \left[\text{Taking out the constant } \frac{4}{a^2}\right]$$

$$= \frac{1}{a^2}\left[2re^{-2r/a} + r^2\left(-\frac{2}{a}\right)e^{-2r/a}\right] \qquad \left[\begin{array}{l}\text{By Product Rule}\\ (uv)' = u'v + uv'\end{array}\right]$$

$$= \frac{4re^{-2r/a}}{a^2}\left[2 + r\left(-\frac{2}{a}\right)\right] \qquad \left[\begin{array}{l}\text{Taking out common factor}\\ re^{-2r/a}\end{array}\right]$$

$$= \frac{4re^{-2r/a}}{a^2}\left[\frac{2a-2r}{a}\right] = \frac{4re^{-2r/a}}{a^3}\left[2a-2r\right] = \frac{8re^{-2r/a}}{a^3}\left[a-r\right]$$

For stationary points we need to equate $P'(r) = 0$. Using the above derivation we have

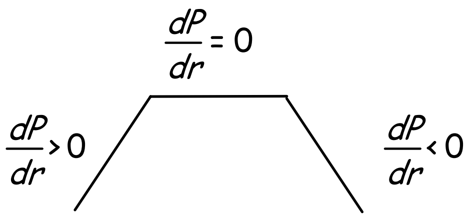
$$P'(r) = \frac{8re^{-2r/a}}{a^3}\left[a-r\right] = 0 \quad \Rightarrow \quad r = 0 \text{ or } r = a$$

Testing what type of stationary point we have at $r = a$ we use the first derivative test:

If $r < a$ then $P'(r) = \frac{8re^{-2r/a}}{a^3}\left[a-r\right] > 0$.

If $r > a$ then $P'(r) = \frac{8re^{-2r/a}}{a^3}\left[a-r\right] < 0$.

This means we have



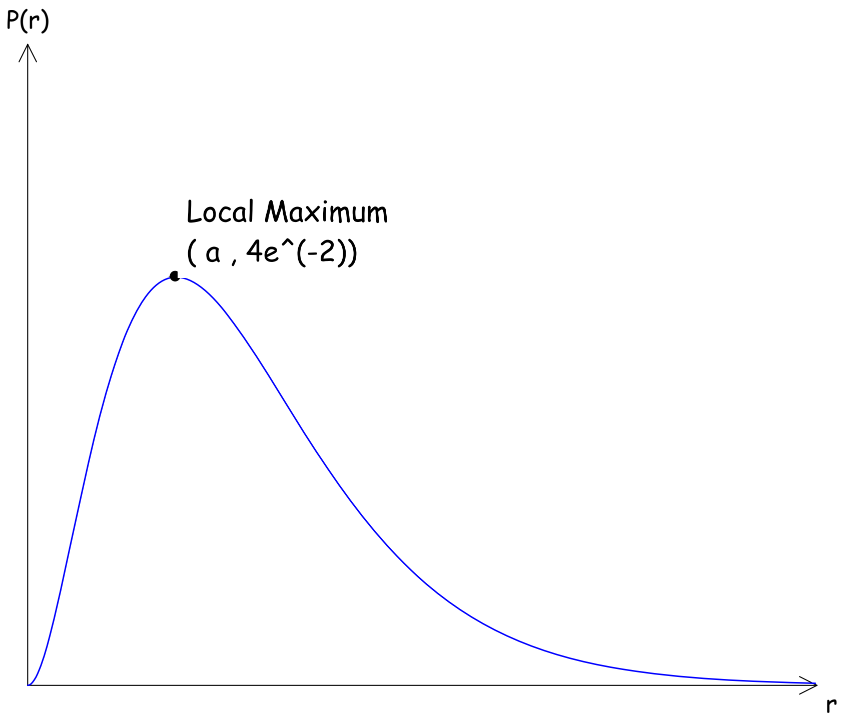
We have a maximum at $r = a$ and the maximum value is found by substituting $r = a$ into $P(r) = \left(\frac{4r^2}{a^2}\right)e^{-2r/a}$. Hence maximum is $P(a) = \left(\frac{4a^2}{a^2}\right)e^{-2a/a} = 4e^{-2}$.

At $r = 0$ we have $P(0) = \left(\frac{4(0)^2}{a^2}\right)e^{-2(0)/a} = 0$. This means that the graph goes through the origin.

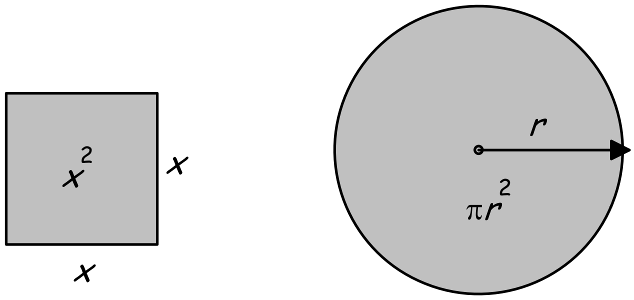
We are given that $r \geq 0$. *What happens to* $P(r)$ *as* $r \to \infty$ *?*

As $r \to \infty$ we have $e^{-2r/a} \to 0$ and therefore $P(r) \to 0$ as $r \to \infty$.

Collecting all the above we have the graph:



16. (a) and (b). Let x be the length of one side of a square and r be the radius of a circle, so that we have:



The total length of the wire is given as 100cm so the perimeter of the square is $4x$ which means that x is between 0 and 25cm. The remaining wire for the circle is $100-4x$.
The perimeter (circumference) of the circle is $2\pi r$ where r is the radius of the circle. From the question we know that the wire is cut to make a square and circle therefore

$$2\pi r = 100-4x$$

$$r = \frac{100-4x}{2\pi} = \frac{50-2x}{\pi} \qquad \left[\begin{array}{l}\text{Dividing numerator and}\\ \text{denominator by 2}\end{array}\right]$$

Let A be the combined area of the circle and square therefore we have

$$A = x^2 + \pi r^2$$

$$= x^2 + \not{\pi}\left(\frac{50-2x}{\not{\pi}}\right)^2 \qquad \left[\text{Substituting } r = \frac{50-2x}{\pi} \text{ from above}\right]$$

$$= x^2 + (2x-50)^2 \qquad \left[\begin{array}{l}\text{Cancelling out } \pi\text{'s and writing}\\ (50-2x)^2 = (-1[-50+2x])^2 = (2x-50)^2\end{array}\right]$$

$$= x^2 + \left(4x^2 - 200x + 50^2\right) \qquad \left[\text{Using } (a-b)^2 = a^2 - 2ab + b^2\right]$$

$$= 5x^2 - 200x + 2500$$

For maximum and minimum we need to differentiate A with respect to x and equate to zero:

$$A = 5x^2 - 200x + 2500$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 10x - 200 = 0$$

Solving the linear equation

$$10x - 200 = 10(x-20) = 0 \quad \Rightarrow \quad x = 20$$

We need to check that this gives maximum or minimum. *How?*
Differentiating again:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 10x - 200$$

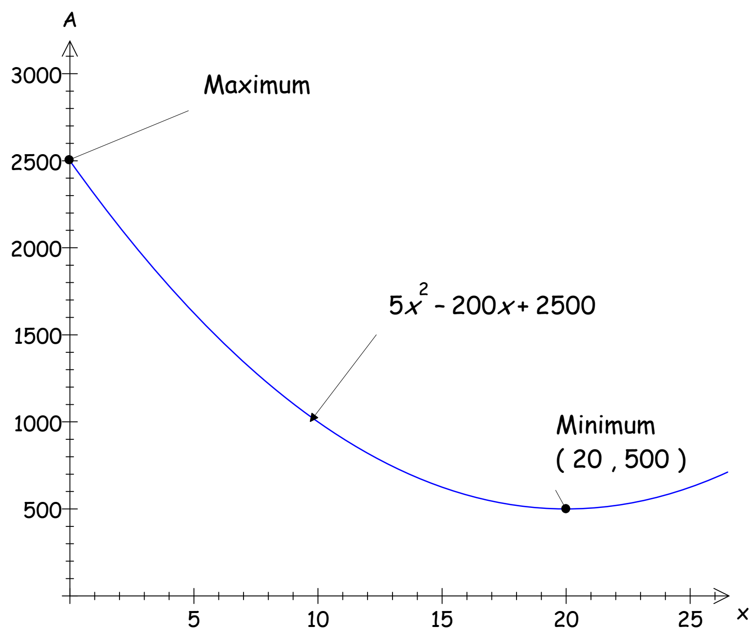
$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 10 > 0 \quad \text{[Positive]}$$

The area is a minimum when we cut a square of length 20cm which means that the wire should be cut at $4 \times 20 = 80$cm in order to make a square.

At what values do we get maximum combined area $A = 5x^2 - 200x + 2500$ *?*

When $x = 0$ which means that we do **not** have a square and the piece is **not** cut but the circle is made of circumference 100cm. *How do we know this?*

We can sketch the graph of the quadratic $A = 5x^2 - 200x + 2500$:



17. *How do we find the radius of convergence of* $\sum_{n=1}^{\infty}\frac{n-1}{(n+2)(2n+5)}\left(\frac{x}{2}\right)^n$ *?*

Use the ratio test. Let $a_n=\frac{n-1}{(n+2)(2n+5)}\left(\frac{x}{2}\right)^n=\frac{(n-1)x^n}{2^n(n+2)(2n+5)}$ then

$$a_{n+1}=\frac{\left[(n+1)-1\right]x^{n+1}}{2^{n+1}\left((n+1)+2\right)\left(2(n+1)+5\right)}=\frac{nx^{n+1}}{2^{n+1}(n+3)(2n+7)}$$

Substituting these into the ratio test formula $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ we have

$$\begin{aligned}
L&=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\\
&=\lim_{n\to\infty}\left|\frac{nx^{n+1}}{2^{n+1}(n+3)(2n+7)}\times\frac{2^n(n+2)(2n+5)}{(n-1)x^n}\right|\\
&=\lim_{n\to\infty}\left|\frac{n\,x^nx}{2^n2\left(2n^2+13n+21\right)}\times\frac{2^n\left(2n^2+9n+10\right)}{(n-1)x^n}\right| && \left[\text{Expanding brackets}\right]\\
&=\lim_{n\to\infty}\left|\frac{x}{2}\frac{n}{\left(2n^2+13n+21\right)}\times\frac{\left(2n^2+9n+10\right)}{(n-1)}\right| && \left[\begin{array}{l}\text{Cancelling common factors}\\ 2^n\ \text{ and }\ x^n\end{array}\right]\\
&=\lim_{n\to\infty}\left|\frac{x}{2}\frac{\left(2n^2+9n+10\right)}{\left(2n^2+13n+21\right)}\times\frac{n}{(n-1)}\right| && (\dagger)
\end{aligned}$$

We can split our rational function as follows and evaluate the limit of each function:

$$\lim_{n\to\infty}\left(\frac{2n^2+9n+10}{2n^2+13n+21}\right)=\lim_{n\to\infty}\left(\frac{2+9/n+10/n^2}{2+13/n+21/n^2}\right)\qquad\left[\begin{array}{l}\text{Dividing numerator}\\ \text{and denominator by } n^2\end{array}\right]$$

$$=\left(\frac{2+0+0}{2+0+0}\right)=1$$

Similarly we have

$$\lim_{n\to\infty}\left(\frac{n}{n-1}\right)=\lim_{n\to\infty}\left(\frac{1}{1-1/n}\right)\qquad\left[\begin{array}{l}\text{Dividing numerator}\\ \text{and denominator by } n\end{array}\right]$$

$$=\left(\frac{1}{1-0}\right)=1$$

Substituting these into (†) gives

$$L=\lim_{n\to\infty}\left|\frac{x}{2}(1)(1)\right|=\left|\frac{x}{2}\right|$$

The power series converges for $L=\left|\frac{x}{2}\right|<1$ which implies that $|x|<2$. Hence the radius of convergence R is equal to 2.