Complete Solutions to Examination Questions 8

1. (a) We need to find $\int (3x - \sin(x))\, dx$.

$$\int (3x - \sin(x))\, dx = \int 3x\, dx - \int \sin(x)\, dx \qquad [\text{Separating integrand}]$$

$$= \frac{3x^2}{2} - [-\cos(x)] + C = \frac{3x^2}{2} + \cos(x) + C$$

(b) We have to find the definite integral $\int_0^1 e^{3x}\, dx$. *How?*

Remember $\int e^{kx} dx = \frac{e^{kx}}{k}$. Using this we have

$$\int_0^1 e^{3x}\, dx = \left[\frac{e^{3x}}{3}\right]_0^1 = \frac{e^3 - e^0}{3} = \frac{e^3 - 1}{3} = 6.36 \ \ (2\text{dp})$$

2. (a) We are given $\int 5x\cos(2x)\, dx$ and taking out 5 gives:

$$\int 5x\cos(2x)\, dx = 5\int x\cos(2x)\, dx$$

The integration by parts formula is (8.45) $\int uv'dx = uv - \int u'vdx$. Let

$$u = x \qquad v' = \cos(2x)$$

$$u' = 1 \qquad v = \int \cos(2x) dx = \frac{\sin(2x)}{2} \qquad \left[\text{By } \int \cos(kx) dx = \frac{\sin(kx)}{k}\right]$$

Substituting these into the integration by parts formula yields

$$5\int x\cos(2x)\, dx = 5\left[uv - \int u'vdx\right]$$

$$= 5\left[x\frac{\sin(2x)}{2} - \int (1)\frac{\sin(2x)}{2} dx\right]$$

$$= 5\left[\frac{x\sin(2x)}{2} - \frac{1}{2}\left(\frac{-\cos(2x)}{2}\right)\right] + C \qquad \left[\text{By } \int \sin(kx) dx = -\frac{\cos(kx)}{k}\right]$$

$$= 5\left[\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right] + C = \frac{5}{4}[2x\sin(2x) + \cos(2x)] + C$$

Our answer is $\frac{5}{4}[2x\sin(2x) + \cos(2x)] + C$.

(b) *Which substitution should we use to find* $\int_1^2 x(x^2+5)^3 dx$ *?*

Let $u = x^2 + 5$ then $\frac{\mathrm{du}}{\mathrm{dx}} = 2x$ and we have $\frac{\mathrm{du}}{2x} = \mathrm{dx}$. We also need to change the limits of integration:

When $x = 1$, $\quad u = 1^2 + 5 = 6 \quad$ and $\quad x = 2$, $\quad u = 2^2 + 5 = 9$. We have

$$\int_1^2 x\left(x^2+5\right)^3 dx = \int_6^9 \not{x} u^3 \frac{du}{2\not{x}}$$

$$= \frac{1}{2}\int_6^9 u^3 du = \frac{1}{2}\left[\frac{u^4}{4}\right]_6^9 = \frac{1}{8}\left(9^4-6^4\right) = \frac{5265}{8}$$

The integral is $\frac{5265}{8}$.

3. *How do we find* $\int\left(\frac{5\sin(2x)}{\cos(x)}\right)dx$ *?*

Use a trigonometric identity for $\sin(2x)$. *Which one?*

$$\sin(2x) = 2\sin(x)\cos(x)$$

Substituting this into the given integrand we have

$$\int\left(\frac{5\sin(2x)}{\cos(x)}\right)dx = \int\left(\frac{5\left[2\sin(x)\cos(x)\right]}{\cos(x)}\right)dx$$

$$= 10\int \sin(x)\,dx \qquad \left[\text{Cancelling } \cos(x) \text{ and taking out } 5\times 2 = 10\right]$$

$$= -10\cos(x) + C$$

4. We can find $\int \frac{\left[\ln(x)\right]^3}{x}dx$ by using substitution. Let $u = \ln(x)$ then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \quad \Rightarrow \quad dx = x\,du$$

Substituting these, $u = \ln(x)$ and $dx = x\,du$, into the given integral we have

$$\int \frac{\left[\ln(x)\right]^3}{x}dx = \int \frac{u^3}{x}x\,du$$

$$= \int u^3\,du \qquad \left[\text{Cancelling out } x's\right]$$

$$= \frac{u^4}{4} + C = \frac{\left[\ln(x)\right]^4}{4} + C$$

5. (i) We can separate out each of the terms in the integrand for $\int_{-1}^{2}\left(4x^3 - x + 1\right)dx$.

$$\int_{-1}^{2}\left(4x^3 - x + 1\right)dx = \left[x^4 - \frac{x^2}{2} + x\right]_{-1}^{2}$$

$$= \left[2^4 - \frac{2^2}{2} + 2\right] - \left[(-1)^4 - \frac{(-1)^2}{2} + (-1)\right] = [16] - \left[-\frac{1}{2}\right] = 16\frac{1}{2}$$

(ii) *How do we find* $\int_0^{\pi/4} \cos\left(2t - \frac{\pi}{4}\right) dt$ *?*

Use $\int \cos(kt+m)\,dt = \frac{\sin(kt+m)}{k}$. By applying this rule we have

$$\int_0^{\pi/4} \cos\left(2t - \frac{\pi}{4}\right) dt = \left[\frac{1}{2}\sin\left(2t - \frac{\pi}{4}\right)\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sin\left(2\frac{\pi}{4} - \frac{\pi}{4}\right) - \sin\left(0 - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{2}\left[\frac{2}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}$$

(iii) Differentiating the denominator $1+x^2$ gives $2x$ therefore

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right) + C$$

(iv) *How do we determine* $\int \theta \sin(\theta)\,d\theta$ *?*

Use integration by parts formula $\int uv'dx = uv - \int u'vdx$. Let

$$u = \theta \qquad v' = \sin(\theta)$$

$$u' = 1 \qquad v = \int \sin(\theta)\,d\theta = -\cos(\theta)$$

Putting these into the integration by parts formula $\int uv'dx = uv - \int u'vdx$ yields

$$\int \theta\sin(\theta)\,d\theta = uv - \int u'v\,d\theta$$

$$= \theta\left[-\cos(\theta)\right] - \int (1)\left(-\cos(\theta)\right) d\theta$$

$$= -\theta\cos(\theta) + \int \cos(\theta)\,d\theta = -\theta\cos(\theta) + \sin(\theta) + C$$

(v) *How do we find* $\int \frac{3}{(x+1)(x-2)}\,dx$ *?*

Use integration by partial fractions. We first write the integrand in partial fractions:

$$\frac{3}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \qquad (\dagger)$$

Multiplying this through by $(x+1)(x-2)$ gives

$$3 = A(x-2) + B(x+1) \qquad (*)$$

Substituting $x=2$ into (*) yields

$$3 = 0 + B(2+1) \quad \Rightarrow \quad B = 1$$

Substituting $x=-1$ into (*) gives

$$3 = A(-1-2) + 0 \quad \Rightarrow \quad A = -1$$

Substituting $A=-1$ and $B=1$ into the integrand $(\dagger)$:

$$\frac{3}{(x+1)(x-2)} = \frac{-1}{x+1} + \frac{1}{x-2} = \frac{1}{x-2} - \frac{1}{x+1}$$ [Writing the positive term first]

We have

$$\int \frac{3}{(x+1)(x-2)}\,dx = \int\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx$$

$$= \int \frac{1}{x-2}\,dx - \int \frac{1}{x+1}\,dx \qquad \text{[Separating the integrand]}$$

$$= \ln|x-2| - \ln|x+1| + C \qquad \left[\text{By (8.42)} \quad \int \frac{f'}{f}\,dx = \ln|f(x)|\right]$$

$$= \ln\left|\frac{x-2}{x+1}\right| + C \qquad \left[\text{Using } \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)\right]$$

(vi) *How do we find* $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$ *?*

Use the integration table to find the appropriate integral. We use

(8.25) $\quad \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right)$

Therefore we have

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}(x)\right]_0^{1/2} = \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] = \frac{\pi}{6}$$

6. *How do we determine* $I = \int_3^4 \frac{9x^2}{(x+1)^2(2x-1)}\,dx$ *?*

Use partial fractions. Converting the integrand into partial fractions we have

$$\frac{9x^2}{(x+1)^2(2x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1} \qquad (*)$$

Multiplying through by $(x+1)^2(2x-1)$ gives

$$9x^2 = A(x+1)(2x-1) + B(2x-1) + C(x+1)^2 \qquad (\dagger)$$

Putting $x = -1$ into (†) gives

$$9(-1)^2 = A(0) + B(2(-1)-1) + C(0)$$

$$9 = -3B \quad \Rightarrow \quad B = -3$$

Putting $x = \frac{1}{2}$ into (†) gives

$$9\left(\frac{1}{2}\right)^2 = A(0) + B(0) + C\left(\frac{1}{2}+1\right)^2$$

$$\frac{9}{4} = \frac{9}{4}C \quad \Rightarrow \quad C = 1$$

How can we find the constant A?

Equate coefficients of x^2 in (†):

$$9 = 2A + C = 2A + 1 \quad \Rightarrow \quad A = 4$$

Substituting $A = 4$, $B = -3$ and $C = 1$ into (*) yields

$$\frac{9x^2}{(x+1)^2(2x-1)} = \frac{4}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{2x-1}$$

Integrating this

$$\int \frac{9x^2}{(x+1)^2(2x-1)}\,dx = \int \frac{4}{x+1}dx - \int \frac{3}{(x+1)^2}\,dx + \int \frac{1}{2x-1}\,dx$$
$$= 4\ln|x+1| - 3\int (x+1)^{-2}\,dx + \frac{1}{2}\ln|2x-1|$$
$$= 4\ln|x+1| + 3(x+1)^{-1} + \frac{1}{2}\ln|2x-1| + C$$
$$= 4\ln|x+1| + \frac{1}{2}\ln|2x-1| + \frac{3}{x+1} + C$$

We have

$$I = \int_3^4 \frac{9x^2}{(x+1)^2(2x-1)}\,dx = \left[4\ln|x+1| + \frac{1}{2}\ln|2x-1| + \frac{3}{x+1}\right]_3^4$$
$$= \left[4\ln|4+1| + \frac{1}{2}\ln|2(4)-1| + \frac{3}{4+1}\right]$$
$$-\left[4\ln|3+1| + \frac{1}{2}\ln|2(3)-1| + \frac{3}{3+1}\right] \quad \left[\begin{array}{l}\text{Substituting limits} \\ x=4 \text{ and } x=3\end{array}\right]$$
$$= \left[4\ln(5) + \frac{1}{2}\ln(7) + \frac{3}{5}\right] - \left[4\ln(4) + \frac{1}{2}\ln(5) + \frac{3}{4}\right]$$
$$= 4\ln\left(\frac{5}{4}\right) + \frac{1}{2}\ln\left(\frac{7}{5}\right) + \frac{3}{5} - \frac{3}{4} \quad \left[\text{Using } \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)\right]$$
$$= 0.91\,(2\text{ dp})$$

7. (a) *How do we find* $\int \frac{2x^2-10x+17}{x-3}\,dx$ *?*

Carry out long division on the integrand $\dfrac{2x^2-10x+17}{x-3}$:

$$\begin{array}{r|l} & 2x \quad -4 \\ \hline x-3 & 2x^2 - 10x + 17 \\ & -\left(2x^2 - 6x\right) \\ \hline & 0 \quad -4x + 17 \\ & -(-4x + 12) \\ \hline & 0 \quad + \quad 5 \end{array}$$

Hence we have $\dfrac{2x^2-10x+17}{x-3} = 2x - 4 + \dfrac{5}{x-3}$. We integrate this separately:

$$\int \frac{2x^2-10x+17}{x-3}\,dx = \int \left(2x - 4 + \frac{5}{x-3}\right)dx$$
$$= \int 2x\,dx - \int 4\,dx + \int \frac{5}{x-3}\,dx$$
$$= x^2 - 4x + 5\ln|x-3| + C$$

(b) *How can we find* $\int \frac{\cos^2(x)}{\sin(x)}\,dx$ *?*

From the fundamental trigonometric identity $\sin^2(x)+\cos^2(x)=1$ we have

$$\cos^2(x)=1-\sin^2(x)$$

Substituting this into the given integral we have

$$\int \frac{\cos^2(x)}{\sin(x)}\,dx=\int \frac{1-\sin^2(x)}{\sin(x)}\,dx$$

$$=\int \frac{1}{\sin(x)}\,dx-\int \frac{\sin^2(x)}{\sin(x)}\,dx$$

$$=\int \operatorname{cosec}(x)\,dx-\int \sin(x)\,dx \qquad \left[\text{Because } \frac{1}{\sin(x)}=\operatorname{cosec}(x)\right]$$

$$=\underbrace{\ln\left|\operatorname{cosec}(x)-\cot(x)\right|}_{\text{By (8.11)}}+\cos(x)+C$$

[(8.11) $\int \operatorname{cosec}(x)\,dx=\ln\left|\operatorname{cosec}(x)-\cot(x)\right|$]

(c) We use integration by substitution to find $\int x(3x-7)^9\,dx$. Let $u=3x-7$ then

$$\frac{\mathrm{du}}{\mathrm{d}x}=3 \qquad dx=\frac{du}{3}$$

What do we substitute for x?

Transposing $u=3x-7$ to make x the subject we have $x=\frac{u+7}{3}$.

Substituting $u=3x-7$, $dx=\frac{du}{3}$ and $x=\frac{u+7}{3}$ into the given integral $\int x(3x-7)^9\,dx$:

$$\int x(3x-7)^9\,dx=\int\left(\frac{u+7}{3}\right)u^9\,\frac{du}{3}$$

$$=\frac{1}{9}\int(u+7)u^9du \qquad \left[\text{Taking out } \frac{1}{3}\frac{1}{3}=\frac{1}{9}\right]$$

$$=\frac{1}{9}\int\left(u^{10}+7u^9\right)du$$

$$=\frac{1}{9}\left[\frac{u^{11}}{11}+\frac{7u^{10}}{10}\right]+C \qquad [\text{Integrating}]$$

$$=\frac{1}{9}\left[\frac{10u^{11}+77u^{10}}{110}\right]+C$$

$$=\frac{u^{10}}{990}[10u+77]+C$$

$$=\frac{(3x-7)^{10}}{990}\left[10(3x-7)+77\right]+C=\frac{(3x-7)^{10}}{990}[30x+7]+C$$

(d) The given integral $\int x\sec(x)\tan(x)\,dx$ is a lot more challenging than parts (a), (b) and (c). *How can we find* $\int x\sec(x)\tan(x)\,dx$ *?*

Use integration by parts formula (8.45) $\int uv'dx = uv - \int u'vdx$. Let

$u = x \qquad v' = \sec(x)\tan(x)$

$u' = 1 \qquad v = \int \sec(x)\tan(x)\,dx = \sec(x) \qquad$ [Using the differentiation table (6.22)]

Substituting $u = x$, $u' = 1$ and $v = \sec(x)$ into the integration by parts formula we have

$$\int x\sec(x)\tan(x)\,dx = uv - \int u'vdx$$
$$= x\sec(x) - \int (1)\sec(x)\,dx$$
$$= x\sec(x) - \underbrace{\ln\left|\sec(x)+\tan(x)\right|}_{\text{By }(8.10)} + C$$

8. (a) To evaluate $\int_0^{\frac{\pi}{3}} 8\cos^3(x)\sin(x)\,dx$ we first need to integrate and then substitute the limits $\pi/3$ and 0. *Which technique do we use to integrate the given function?*
Substitution because when we differentiate cos we get minus sin. Let

$$u = \cos(x)$$
$$\frac{\mathrm{du}}{\mathrm{d}x} = -\sin(x) \quad \Rightarrow \quad \mathrm{d}x = -\frac{\mathrm{d}u}{\sin(x)}$$

We also need to change the limits:

$x = 0 \qquad u = \cos(0) = 1$

$x = \frac{\pi}{3} \qquad u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Substituting these we have

$$\int_0^{\frac{\pi}{3}} 8\cos^3(x)\sin(x)\,dx = 8\int_0^{\frac{\pi}{3}} \cos^3(x)\sin(x)\,dx$$
$$= 8\int_1^{1/2} u^3\sin(x)\left(\frac{-du}{\sin(x)}\right)$$
$$= -8\int_1^{1/2} u^3 du \qquad \text{[Cancelling } \sin(x)\text{]}$$
$$= -8\left[\frac{u^4}{4}\right]_1^{1/2} = -2\left[\left(\frac{1}{2}\right)^4 - 1\right] = -2\left[-\frac{15}{16}\right] = \frac{15}{8}$$

(b) *How do we evaluate* $\int_1^4 \frac{\ln(x)}{\sqrt{x}}\,dx$ *?*

Use integration by parts formula (8.45) $\int uv'dx = uv - \int u'vdx$. Let

$u = \ln(x) \qquad v' = x^{-1/2} \qquad \left[\text{Because } \frac{1}{\sqrt{x}} = x^{-1/2}\right]$

$u' = \frac{1}{x} \qquad v = \int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} = 2x^{1/2}$

We have

$$\int_1^4 \frac{\ln(x)}{\sqrt{x}}\,dx = [uv]_1^4 - \int_1^4 (u'v)\,dx$$

$$= \left[\ln(x)2x^{1/2}\right]_1^4 - \int_1^4 \frac{1}{x}2x^{1/2}\,dx$$

$$= 2\left[\sqrt{x}\ln(x)\right]_1^4 - 2\int_1^4 x^{-1/2}dx \qquad \left[\text{Because } \frac{1}{x}x^{1/2} = x^{-1}x^{1/2} = x^{-1/2}\right]$$

$$= 2\left[\sqrt{4}\ln(4) - 1\ln(1)\right] - 2\left[\frac{x^{1/2}}{1/2}\right]_1^4$$

$$= 2\left[2\ln(4)\right] - 4\left[\sqrt{4} - \sqrt{1}\right] = 4\left[\ln(4) - 1\right]$$

9. We need to find $\int_0^{\ln(\pi)} e^x \sin\left(e^x\right)dx$. Use integration by substitution. Let

$$u = e^x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = e^x \;\Rightarrow\; \mathrm{d}x = \frac{\mathrm{d}u}{e^x}$$

Changing the limits:

$x = 0 \qquad u = e^0 = 1$

$x = \ln(\pi) \qquad u = e^{\ln(\pi)} = \pi$

Using this substitution we have

$$\int_0^{\ln(\pi)} e^x \sin\left(e^x\right)dx = \int_1^{\pi} e^x \sin(u)\frac{du}{e^x}$$

$$= \int_1^{\pi} \sin(u)\,du \qquad \left[\text{Cancelling } e^x\right]$$

$$= -\left[\cos(u)\right]_1^{\pi}$$

$$= -\left[\cos(\pi) - \cos(1)\right] = -\left[-1 - 0.5403\right] = 1.54\ (2\text{ dp})$$

10. *How do we integrate* $\ln(x)$*?*

Use integration by parts formula with $u = \ln(x)$ and $v' = 1$ because $\ln(x) = \ln(x)\times 1$.

$$u' = \frac{1}{x} \text{ and } v = \int 1\,dx = x$$

Substituting these into the integration by parts formula (8.45) $\int uv'dx = uv - \int(u'v)dx$:

$$\int_1^e \ln(x)\,dx = [uv]_1^e - \int_1^e (u'v)\,dx$$

$$= \left[x\ln(x)\right]_1^e - \int_1^e \left(\frac{1}{x}x\right)dx \qquad \left[\text{Because } u = \ln(x),\ u' = \frac{1}{x} \text{ and } v = x\right]$$

$$= \left[e\ln(e) - 0\right] - \int_1^e (1)\,dx = e\underbrace{\ln(e)}_{=1} - [x]_1^e = e - [e-1] = 1$$

11. We apply partial fractions to find $\int_3^5 \frac{t+5}{(t-1)(t+2)}\,dt$. Converting the integrand into partial fractions we have

$$\frac{t+5}{(t-1)(t+2)} = \frac{A}{t-1} + \frac{B}{t+2} \qquad (☼)$$

Multiplying this by $(t-1)(t+2)$ gives

$$t+5 = A(t+2) + B(t-1) \qquad (*)$$

Substituting $t=1$ into (*) gives

$$6 = 3A \quad \Rightarrow \quad A = 2$$

Substituting $t=-2$ into (*) gives

$$3 = -3B \quad \Rightarrow \quad B = -1$$

Substituting $A=2$ and $B=-1$ into (☼)

$$\frac{t+5}{(t-1)(t+2)} = \frac{2}{t-1} - \frac{1}{t+2}$$

We can evaluate our given integral as follows:

$$\int_3^5 \frac{t+5}{(t-1)(t+2)}\,dt = \int_3^5 \frac{2}{t-1}\,dt - \int_3^5 \frac{1}{t+2}\,dt$$

$$= 2\left[\ln(t-1)\right]_3^5 - \left[\ln(t+2)\right]_3^5 \qquad \left[\text{Using } \int \frac{f'}{f}\,dx = \ln|f|\right]$$

$$= 2\left[\ln(4) - \ln(2)\right] - \left[\ln(7) - \ln(5)\right]$$

$$= 2\ln\left(\frac{4}{2}\right) - \ln\left(\frac{7}{5}\right) \qquad \left[\text{Using } \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)\right]$$

$$= 2\ln(2) - \ln\left(\frac{7}{5}\right)$$

$$= \ln(4) - \ln(1.4) = \ln\left(\frac{4}{1.4}\right)$$

12. (i) We need to determine $\int_1^2 e^{x-1}dx$.

$$\int_1^2 e^{x-1}dx \underbrace{=}_{\text{Integrating}} \left[e^{x-1}\right]_1^2 = e^1 - e^0 = e - 1$$

(ii) *How do we find* $\int_{-1}^1 x^2 e^x dx$ *?*

Need to use integration by parts twice because we have an x^2 term. We first determine the indefinite integral and then put in the limits to find the given definite integral. Let

$$u = x^2 \qquad v' = e^x$$

$$u' = 2x \qquad v = \int e^x dx = e^x$$

Substituting these into the integration by parts formula (8.45) $\int uv'dx = uv - \int u'vdx$:

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx = x^2 e^x - 2\int xe^x dx \qquad (\dagger)$$

We need to find the last integral $\int xe^x dx$ on the right hand side of $(\dagger)$. *How?*
Use integration by parts again. Let

$$u = x \qquad v' = e^x$$
$$u' = 1 \qquad v = e^x$$

We have

$$\int xe^x dx = uv - \int u'v dx$$
$$= xe^x - \int e^x dx = xe^x - e^x + C$$

Putting this into (†) gives

$$\int x^2 e^x dx = x^2 e^x - 2\left[xe^x - e^x\right] + C$$
$$= \left(x^2 - 2x + 2\right)e^x + C$$

Evaluating the given definite integral by substituting in the limits:

$$\int_{-1}^{1} x^2 e^x dx = \left[\left(x^2 - 2x + 2\right)e^x\right]_{-1}^{1}$$
$$= \left[\left(1^2 - (2\times 1) + 2\right)e^1\right] - \left[\left((-1)^2 - (2\times(-1)) + 2\right)e^{-1}\right]$$
$$= e - 5e^{-1}$$

Our final answer is $e - 5e^{-1}$.

13. We first find $F(x) = \int_x^{x^2} \frac{1}{\sqrt{1+t^2}} dt$. *How?*

$$F(x) = \int_x^{x^2} \frac{1}{\sqrt{1+t^2}} dt = \int_{t=x}^{t=x^2} \frac{1}{\sqrt{1+t^2}} dt$$
$$= \int \frac{1}{\sqrt{1+\left(x^2\right)^2}} d\left(x^2\right) - \int \frac{1}{\sqrt{1+(x)^2}} dx$$
$$= \int \frac{2x}{\sqrt{1+\left(x^2\right)^2}} dx - \int \frac{1}{\sqrt{1+(x)^2}} dx$$

We know from the Fundamental Theorem of Calculus that differentiation and integration are inverses processes therefore

$$F'(x) = \frac{2x}{\sqrt{1+\left(x^2\right)^2}} - \frac{1}{\sqrt{1+(x)^2}}$$
$$= \frac{2x}{\sqrt{1+x^4}} - \frac{1}{\sqrt{1+x^2}}$$

Hence

$$F'(-1) = \frac{2(-1)}{\sqrt{1+(-1)^4}} - \frac{1}{\sqrt{1+(-1)^2}} = -\frac{3}{\sqrt{2}}$$

14. We are told to use the substitution $x = 2\tan(\theta)$. We need to differentiate this to find $\mathrm{d}x$.

$$x = 2\tan(\theta)$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec^2(\theta) \qquad dx = 2\sec^2(\theta)\,\mathrm{d}\theta$$

Using this substitution to find $I(x) = \int \frac{1}{x^2\sqrt{4+x^2}}\,dx$:

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx = \int \frac{1}{4\tan^2(\theta)\sqrt{4+4\tan^2(\theta)}} 2\sec^2(\theta)\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\tan^2(\theta)\,\cancel{2}\sqrt{\sec^2(\theta)}}\cancel{2}\sec^2(\theta)\,d\theta \qquad \left[\text{Because } 1+\tan^2(\theta) = \sec^2(\theta)\right]$$

$$= \frac{1}{4}\int \frac{1}{\tan^2(\theta)\sec(\theta)}\sec^2(\theta)\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\tan^2(\theta)}\sec(\theta)\,d\theta \qquad \left[\text{Cancelling } \sec(\theta)\text{'}s\right]$$

$$= \frac{1}{4}\int \frac{\cos^2(\theta)}{\sin^2(\theta)}\frac{1}{\cos(\theta)}\,d\theta = \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = I(\theta)$$

We are told to use the substitution $u = \sin(\theta)$ on this $I(\theta) = \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta$. Differentiating u we have

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \cos(\theta) \qquad \Rightarrow \qquad d\theta = \frac{du}{\cos(\theta)}$$

Using this we have

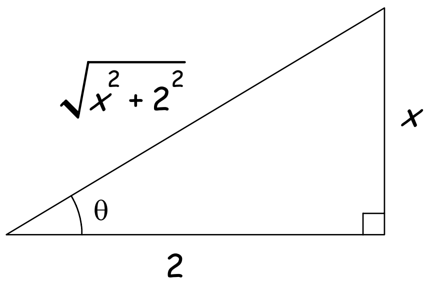
$$I(\theta) = \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos(\theta)}{u^2}\frac{du}{\cos(\theta)} = \frac{1}{4}\int u^{-2}du = \frac{1}{4}\left[-u^{-1}\right] + c = -\frac{1}{4u} + c = I(u)$$

This is our required result. Using $u = \sin(\theta)$ we have

$$I(\theta) = -\frac{1}{4\sin(\theta)} + c \qquad (*)$$

We need to find $\sin(\theta)$ in terms of *x*. We are given $x = 2\tan(\theta)$ and considering the right-angled triangle with the opposite side equal to *x* and adjacent side equal to 2.



What is $u = \sin(\theta)$ *equal to?*

$$\sin(\theta)=\frac{opp}{hyp}=\frac{x}{\sqrt{x^2+4}}$$

Substituting this into (*) gives

$$I(x)=-\frac{1}{4\frac{x}{\sqrt{x^2+4}}}+c=-\frac{\sqrt{x^2+4}}{4x}+c$$

This is our final required result.

15. We need to determine $\int\cos\left(\sqrt{x}\right)dx$. *Which substitution should we use?*

Let $u=\sqrt{x}=x^{1/2}$ then $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$ $\Rightarrow$ $dx=2\sqrt{x}\,\mathrm{d}u=2u\,du$.

Putting this into the given integral we have

$$\int\cos\left(\sqrt{x}\right)dx=\int\cos(u)2u\,du$$

$$=2\int u\cos(u)\,du \qquad (*)$$

How do we integrate $\int u\cos(u)\,du$ *?*

Use integration by parts formula with the symbols *p* and *v* because we have already used *u* in the above substitution. Let

$$p=u \qquad v'=\cos(u)$$

$$p'=1 \qquad v=\int\cos(u)\,du=\sin(u)$$

Substituting these into the integration by parts formula gives

$$\int u\cos(u)\,du=pv-\int p'vdu$$

$$=u\sin(u)-\int\sin(u)\,du$$

$$=u\sin(u)-\left[-\cos(u)\right]+C=u\sin(u)+\cos(u)+C$$

Remember $u=\sqrt{x}$. Substituting this and the above into (*) gives

$$\int\cos\left(\sqrt{x}\right)dx=2\left[u\sin(u)+\cos(u)\right]+C$$

$$=2\left[\sqrt{x}\sin\left(\sqrt{x}\right)+\cos\left(\sqrt{x}\right)\right]+C$$

16. We need to find $\int e^{\sqrt{x}}\frac{dx}{\sqrt{x}}$. *Which method do we use to find this integral?*

Use the substitution $u=\sqrt{x}=x^{1/2}$. Differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}} \quad \text{implies that } \mathrm{d}x=2\sqrt{x}\,\mathrm{d}u$$

Hence we have

$$\int e^{\sqrt{x}}\frac{dx}{\sqrt{x}}=\int e^{u}\frac{2\sqrt{x}}{\sqrt{x}}\mathrm{d}u$$

$$=2\int e^{u}du=2e^{u}+C=2e^{\sqrt{x}}+C$$

The given integral is equal to $2e^{\sqrt{x}}+C$.

17. This integral $\int e^{\sqrt{x}}dx$ is a lot more demanding than the integral given in question 16 above. Substitution **might** work, so let us try by letting $u=\sqrt{x}=x^{1/2}$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}} \qquad \Rightarrow \qquad dx=2\sqrt{x}du=2udu$$

Substituting these into the given integral we have

$$\int e^{\sqrt{x}}dx=\int\left(e^{u}2u\right)du=2\int\left(ue^{u}\right)du \qquad (*)$$

How do we determine $\int\left(ue^{u}\right)du$ *?*

Apply the integration by parts formula (8.45) $\int pv'dx=pv-\int p'vdx$:

$$p=u \qquad v'=e^{u}$$

$$p'=1 \qquad v=\int e^{u}du=e^{u}$$

We have

$$2\int\left(ue^{u}\right)du=2\left[pv-\int p'vdx\right]$$

$$=2\left[ue^{u}-\int e^{u}du\right]$$

$$=2\left[ue^{u}-e^{u}\right]+C=2e^{u}\left[u-1\right]+C$$

Remember $u=\sqrt{x}$ so substituting this back into (*) yields

$$\int e^{\sqrt{x}}dx=2e^{u}\left[u-1\right]+C$$

$$=2e^{\sqrt{x}}\left[\sqrt{x}-1\right]+C$$