Complete Solutions to Examination Questions 5

1. We use the exponential definition of hyperbolic functions to evaluate:

$$\sinh(1.5)=\frac{e^{1.5}-e^{-1.5}}{2}=\frac{4.482-0.223}{2}=\frac{4.259}{2}=2.13$$

$$\cosh(0.3)=\frac{e^{0.3}+e^{-0.3}}{2}=\frac{1.35+0.74}{2}=\frac{2.09}{2}=1.045$$

$$\operatorname{sech}(1.2)=\frac{1}{\cosh(1.2)}=\frac{2}{e^{1.2}+e^{-1.2}}=\frac{2}{3.32+0.30}=\frac{2}{3.62}=0.55$$

You may like to check your answers by using the hyperbolic functions of your calculator.

2. (a) Substituting $d=4$ into $P(d)=25e^{0.1d}$ gives $P(4)=25e^{0.1\times 4}=25e^{0.4}=37.3\text{ W}$.
(b) Substituting $d=12$ into $P(d)=25e^{0.1d}$ gives $P(12)=25e^{0.1\times 12}=25e^{1.2}=83\text{ W}$.

3. (i) Using the rule $\log(A)+\log(B)=\log(AB)$ we have

$$\log_2(3)+\log_2(x+1)=\log_2[3(x+1)]=\log_2[3x+3]=\log_2(x+11)$$

We have

$$3x+3=x+11 \Rightarrow 2x=8 \Rightarrow x=4$$

(ii) Dividing the given equation $2\left[\ln\left(x^2+4\right)\right]=3.9$ by 2:

$$\ln\left(x^2+4\right)=1.95$$

How do we remove the natural log, ln, from the Left Hand Side?
Take exponentials because $e^{\ln(u)}=u$. Taking exponentials of both sides gives

$$x^2+4=e^{1.95}$$

$$x^2=e^{1.95}-4=3.03$$

Taking the square root yields $x=\pm\sqrt{3.03}$.
(iii) Dividing the given equation $5\left(2^{6x+1}\right)=42$ by 5:

$$2^{6x+1}=8.4$$

Taking natural logs, ln, of both sides gives

$$\ln\left(2^{6x+1}\right)=\ln(8.4)$$

$$(6x+1)\ln(2)=\ln(8.4) \qquad \left[\text{Using } \ln\left(A^n\right)=n\ln(A)\right]$$

$$6x+1=\frac{\ln(8.4)}{\ln(2)}=3.07$$

Subtracting 1 and dividing by 6 gives

$$x=\frac{3.07-1}{6}=0.345$$

4. We are given that the law connecting p and V is $p=aV^2+bV$. Dividing this law by V yields

$$\frac{p}{V}=aV+b$$

This is a straight line with gradient equal to a and vertical intercept equal to b, that is

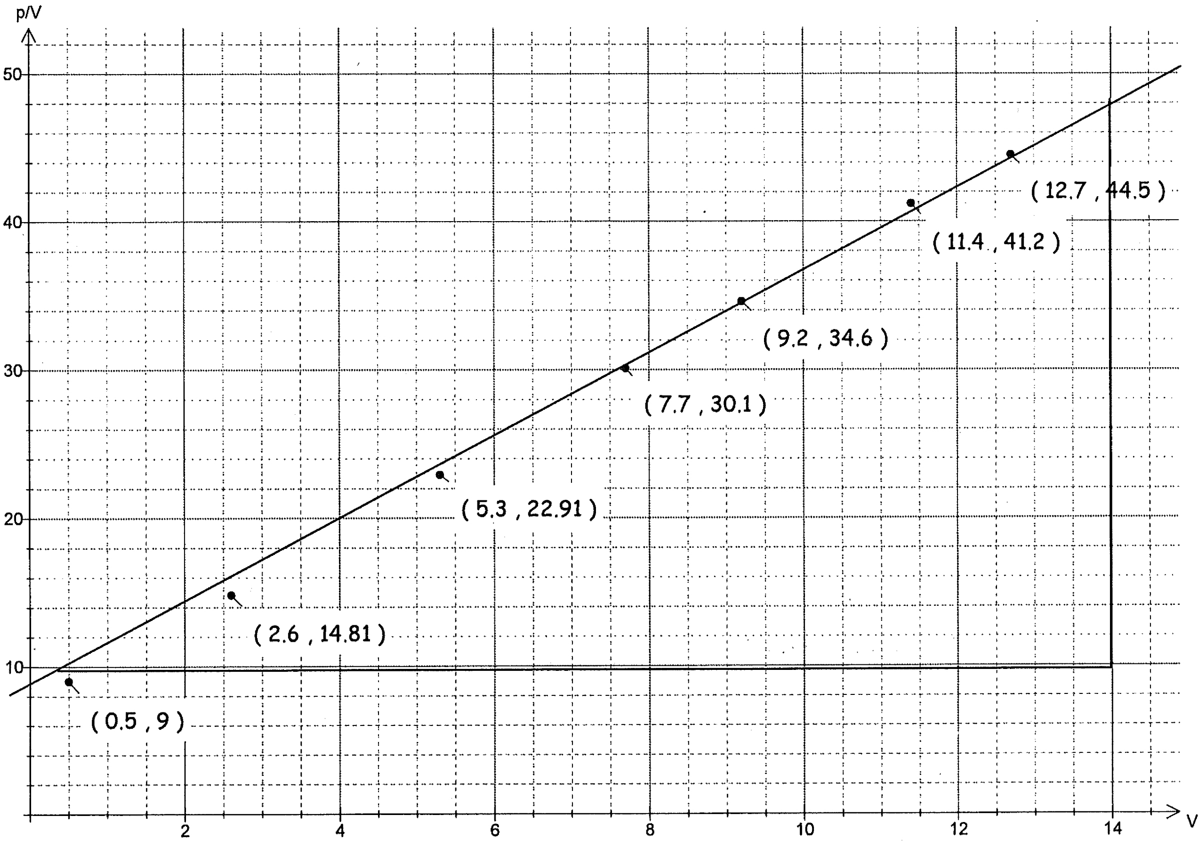
$$\frac{p}{V} = aV + b \leftarrow \text{vertical intercept}$$

$$\uparrow \text{gradient}$$

We need to plot p/V against V. The table of values is given by:

p	4.5	38.5	121.4	231.8	318.2	469.7	565.2
V	0.5	2.6	5.3	7.7	9.2	11.4	12.7
p/V	9	14.81	22.91	30.10	34.59	41.20	44.50

The graph of p/V against V is:



The best fit line is drawn above.

Since we have a straight line in the above therefore the gradient of the line gives the value of a and the vertical intercept gives the value of b. *What is the vertical intercept in the above graph?*

Vertical intercept $= 9$ therefore the value of b is 9. The gradient a is given by

$$\text{gradient} = \frac{48-10}{14-0.5} = \frac{38}{13.5} = 2.815$$

Hence $a = 2.815$. By substituting $a = 2.815$ and $b = 9$ into the given equation we conclude that the table of values obeys the law

$$p = aV^2 + bV = 2.815V^2 + 9V$$

5. (i) We need to solve $8^x = \frac{1}{2}$. *How?*

Since $2^3 = 8$ and $2^{-1} = \frac{1}{2}$ therefore we have

$$\left(2^3\right)^x = 2^{-1}$$

$$2^{3x} = 2^{-1}$$

Taking both sides to the log base 2 we have $3x = -1 \quad \Rightarrow \quad x = -\frac{1}{3}$

(ii) *How do we find x in* $\log_4 x = 2$ *?*

By definition of logs. This means that $x = 4^2 = 16$.

6. (i) *How do we solve* $8^x = 2$ *?*

Since $2^3 = 8$ therefore $2 = 8^{1/3}$ which means that $x = 1/3$.

(ii) *What does the given equation* $\log_3\left(\frac{27}{x+1}\right) = 2$ *mean with respect to indices?*

Means that $\frac{27}{x+1} = 3^2 = 9$. Rearranging this equation gives

$$27 = 9(x+1) = 9x + 9$$

$$9x = 27 - 9 = 18 \quad \Rightarrow \quad x = 2$$

(iii) Similarly for this $\log_2\left(4^{x+1}\right) = x$ we have

$$4^{x+1} = 2^x$$

Writing $4 = 2^2$ we have $\left(2^2\right)^{x+1} = 2^x$ implies $2^{2x+2} = 2^x$. Hence

$$2x + 2 = x$$

$$x = -2$$

(iv) Writing the given logarithm $\log_x\left(x^3 - 2x + 1\right) = 3$ in terms of indices we have

$$x^3 - 2x + 1 = x^3$$

$$2x - 1 = 0 \quad \text{implies } x = \frac{1}{2}$$

7. *How do we solve* $9^{3-2x} = 3 \cdot 27^{x-3}$ *?*

Since $9 = 3^2$ and $27 = 3^3$ we have

$$\left(3^2\right)^{3-2x} = 3 \cdot \left(3^3\right)^{x-3}$$

Applying the rules of indices $\left(a^m\right)^n = a^{mn}$ gives

$$3^{2(3-2x)} = 3 \cdot 3^{3(x-3)}$$

$$3^{6-4x} = 3 \cdot 3^{3x-9} = 3^{3x-9+1} = 3^{3x-8}$$

Taking both sides to log 3 we have

$$6 - 4x = 3x - 8$$

$$14 = 7x \quad \text{implies } x = 2$$

8. a) Writing $\log\left(\frac{A^2\sqrt{C}}{B^5}\right)$ in terms of log A, log B and log C:

$$\log\left(\frac{A^2\sqrt{C}}{B^5}\right)=\log\left(A^2\sqrt{C}\right)-\log\left(B^5\right) \qquad \left[\text{Using } \log\left(\frac{X}{Y}\right)=\log(X)-\log(Y)\right]$$

$$=\log\left(A^2\right)+\log\left(\sqrt{C}\right)-\log\left(B^5\right) \qquad \left[\text{Using } \log(XY)=\log(X)+\log(Y)\right]$$

$$=2\log(A)+\frac{1}{2}\log(C)-5\log(B) \qquad \left[\text{Using } \log\left(X^n\right)=n\log(X)\right]$$

b) *How do we solve the equation* $3^{2x-1}=18$*?*
Taking natural logs of both sides:

$$\ln\left(3^{2x-1}\right)=\ln(18)$$

$$(2x-1)\ln(3)=\ln(18)$$

$$2x-1=\frac{\ln(18)}{\ln(3)}=2.63$$

Adding 1 and dividing by 2 yields

$$x=\frac{2.63+1}{2}=1.815$$

c) (i) Substituting $x=2$ into $y=\ln\left(x^2-3\right)$ gives $y=\ln\left(2^2-3\right)=\ln(1)=0$.

(ii) Substituting $y=1$ into $y=\ln\left(x^2-3\right)$ gives

$$\ln\left(x^2-3\right)=1$$

Taking exponential of both sides

$$x^2-3=e^1=2.718$$

$$x^2=2.718+3=5.718$$

$$x=\pm\sqrt{5.718}=\pm 2.39$$

9. i) By substituting the given values $I=2$ at $v=0.003$ into $I=I_S\left(e^{40v}-1\right)$:

$$2=I_S\underbrace{\left(e^{40\times 0.003}-1\right)}_{=0.1275}=0.1275 I_S \quad \Rightarrow \quad I_s=\frac{2}{0.1275}=15.686$$

ii) Need to find the value of v when $I=3$. *How?*
By substituting $I_s=15.686$ and $I=3$ into the given formula $I=I_S\left(e^{40v}-1\right)$:

$$15.686\left(e^{40v}-1\right)=3$$

$$e^{40v}-1=\frac{3}{15.686}=0.191$$

We have $e^{40v}=0.191+1=1.191$. Taking natural logs of both sides yields:

$$\ln\left(e^{40v}\right)=\ln(1.191)$$

$$40v\underbrace{\ln(e)}_{=1}=0.175 \quad \text{implies} \quad v=\frac{0.175}{40}=4.375\times 10^{-3}$$

We have $v=4.375\times 10^{-3}$.

10. By applying the formula for the difference of two squares and writing $e^{2y}=\left(e^{y}\right)^{2}$:

$$\lim_{y\to 0}\frac{\left(e^{2y}-1\right)}{\left(e^{y}-1\right)}=\lim_{y\to 0}\frac{\left(e^{y}\right)^{2}-1}{e^{y}-1}$$

$$=\lim_{y\to 0}\frac{\left[e^{y}-1\right]\left[e^{y}+1\right]}{e^{y}-1}\qquad\left[\begin{array}{l}\text{Using } a^{2}-b^{2}=(a-b)(a+b)\\ \text{on numerator}\end{array}\right]$$

$$=\lim_{y\to 0}\left[e^{y}+1\right]\qquad\left[\text{Cancelling } e^{y}-1\right]$$

$$=\lim_{y\to 0}\left[e^{y}\right]+1=1+1=2$$

11. By the standard formula for tanh we have $\tanh(x)=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$. This can be rewritten as:

$$\tanh(x)=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{e^{x}-\dfrac{1}{e^{x}}}{e^{x}+\dfrac{1}{e^{x}}}\qquad\left[\text{Because } e^{-x}=\frac{1}{e^{x}}\right]$$

$$=\frac{e^{x}e^{x}-1}{e^{x}e^{x}+1}=\frac{e^{2x}-1}{e^{2x}+1}$$

Replacing the x with $1/x$ yields:

$$\tanh\left(\frac{1}{x}\right)=\frac{e^{2/x}-1}{e^{2/x}+1}$$

The given equation $\left(1-e^{2/x}\right)/\left(1+e^{2/x}\right)$ is the above with a minus sign:

$$\frac{1-e^{2/x}}{1+e^{2/x}}=\frac{-\left(e^{2/x}-1\right)}{e^{2/x}+1}=-\tanh\left(\frac{1}{x}\right)$$

12. a. Substituting $P(t)=0.6$ into $P(t)=\dfrac{0.9}{1+6e^{-0.32t}}$ gives

$$\frac{0.9}{1+6e^{-0.32t}}=0.6$$

What do we need to find?
The value of t. Rearranging the above equation yields

$$\frac{0.9}{0.6}=1+6e^{-0.32t}\quad\text{which means that}\quad 1+6e^{-0.32t}=1.5$$

Subtract 1 and divide by 6 gives $e^{-0.32t}=\dfrac{1.5-1}{6}=8.33\times 10^{-2}$. *How do we find t?*

Take natural logs of both sides:

$$\ln\left(e^{-0.32t}\right)=\ln\left(8.333\times 10^{-2}\right)$$

$$-0.32t=-2.485\quad\text{implies}\quad t=7.77$$

This means that 60% households will have a DVD player by the end of the year 2007.

b. Similarly we substitute $P(t)=0.8$ into $P(t)=\dfrac{0.9}{1+6e^{-0.32t}}$ gives

$$\frac{0.9}{1+6e^{-0.32t}}=0.8$$

What do we need to find?
The value of *t*. Rearranging the above equation yields

$$\frac{0.9}{0.8}=1+6e^{-0.32t} \text{ which means that } 1+6e^{-0.32t}=1.125$$

Subtract 1 and divide by 6 gives $e^{-0.32t}=\frac{1.125-1}{6}=2.1\times10^{-2}$. Take logs of both sides:

$$-0.32t=\ln\left(2.1\times10^{-2}\right) \text{ implies } t=\frac{\ln\left(2.1\times10^{-2}\right)}{-0.32}=12.07$$

This means that 80% households will have a DVD player by the beginning of the year 2012.

13. a) See solution to Example 21 in the text for $1+\sinh^2 x=\cosh^2 x$.
b) Required to show that $\cosh 2x=1+2\sinh^2 x$. *How?*
Substituting the exponential definition of sinh and expanding:

$$1+2\sinh^2 x=1+2\left(\frac{e^x-e^{-x}}{2}\right)^2 \qquad \left[\text{Because } \sinh(x)=\frac{e^x-e^{-x}}{2}\right]$$

$$=1+2\left(\frac{e^{2x}-2e^{-x}e^x+e^{-2x}}{4}\right) \qquad \left[\text{Using } (a-b)^2=a^2-2ab+b^2\right]$$

$$=1+\left(\frac{e^{2x}-2+e^{-2x}}{2}\right) \qquad \left[\text{Because } e^{-x}e^x=1 \text{ and } \frac{2}{4}=\frac{1}{2}\right]$$

$$=\frac{2+e^{2x}-2+e^{-2x}}{2}=\frac{e^{2x}+e^{-2x}}{2}=\cosh 2x \qquad \left[\text{Because } \frac{e^x+e^{-x}}{2}=\cosh(x)\right]$$

14. To find each of the hyperbolic functions we need to use the hyperbolic identities.
(a) *How is* $\cosh(x)$ *related to* $\sinh(x)$*?*

By (5.32) we have $\cosh^2(x)-\sinh^2(x)=1$. We are given that $\sinh(x)=\frac{12}{5}$ therefore

$$\sinh^2(x)=\left(\frac{12}{5}\right)^2=\frac{144}{25}$$

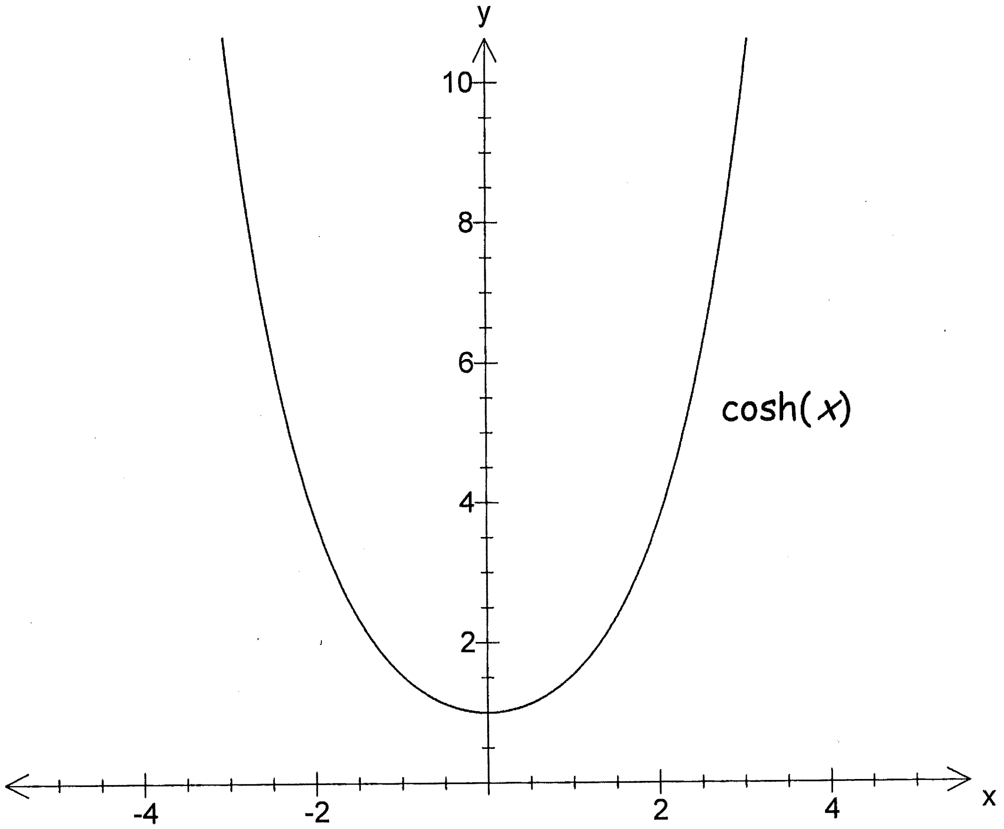
Substituting this into the above identity $\cosh^2(x)-\sinh^2(x)=1$ gives

$$\cosh^2(x)-\frac{144}{25}=1$$

$$\cosh^2(x)=1+\frac{144}{25}=\frac{25+144}{25}=\frac{169}{25}$$

$$\cosh(x)=\sqrt{\frac{169}{25}}=\frac{13}{5}$$

[Remember the cosh function is greater than or equal to 1 as you can observe from the graph:



This means that cosh cannot be negative].

Taking the positive root we have $\cosh(x) = \frac{13}{5}$.

(b) *What identity relates* $\tanh(x)$ *to sinh and cosh?*

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{12/5}{13/5} = \frac{12}{13}$$

(c) *Do you know any identity involving* $\sinh(2x)$*?*

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

Substituting $\sinh(x) = \frac{12}{5}$ and $\cosh(x) = \frac{13}{5}$ into this $\sinh(2x) = 2\sinh(x)\cosh(x)$:

$$\sinh(2x) = 2\frac{12}{5}\frac{13}{5} = \frac{312}{25} = 12.48$$

15. Substituting the exponential definitions of sinh and cosh into $\sinh(x) + 3\cosh(x) = 4$:

$$\sinh(x) + 3\cosh(x) = \left(\frac{e^x - e^{-x}}{2}\right) + 3\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^x - e^{-x} + 3e^x + 3e^{-x}}{2}$$

$$= \frac{4e^x + 2e^{-x}}{2} = 2e^x + e^{-x}$$

Remember all this is equal to 4, so we have $2e^x + e^{-x} = 4$. *How do we solve this equation?*

We know by the rules of indices that $e^{-x} = \frac{1}{e^x}$. Substituting this into the above $2e^x + e^{-x} = 4$ gives

$$2e^x + \frac{1}{e^x} = 4$$

$$2e^x e^x + 1 = 4e^x \qquad \left[\text{Multiplying through by } e^x\right]$$

$$2\left(e^x\right)^2 - 4e^x + 1 = 0$$

How do we solve $2\left(e^x\right)^2 - 4e^x + 1 = 0$*?*

Let $t = e^x$ then we have a quadratic equation:

$$2t^2 - 4t + 1 = 0$$

We can use the quadratic formula $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 2,\ b = -4$ and $c = 1$.

$$t = \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 2 \times 1)}}{2 \times 2}$$

$$= \frac{4 \pm \sqrt{16 - 8}}{4}$$

$$= \frac{4 - 2.828}{4},\ \frac{4 + 2.828}{4}$$

$$= 0.293,\ \ 1.707$$

Since $t = e^x = 0.293,\ \ 1.707$ therefore taking natural logs gives

$$x = \ln(0.293),\ \ \ln(1.707)$$

$$= -1.227,\ \ 0.535$$

The values of x are $-1.227,\ 0.535$.