Complete Solutions to Examination Questions 11

1. (i) *How do we find* $(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$ *given* $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ *and* $\mathbf{B}=\begin{pmatrix}5&6\\7&8\end{pmatrix}$ *?*

Evaluate $\mathbf{A}-\mathbf{B}$, $\mathbf{A}+\mathbf{B}$ and then multiply them together.

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}1&2\\3&4\end{pmatrix}-\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}-4&-4\\-4&-4\end{pmatrix}=-4\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}6&8\\10&12\end{pmatrix}=2\begin{pmatrix}3&4\\5&6\end{pmatrix}$$

Multiplying these together gives

$$(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})=-4\begin{pmatrix}1&1\\1&1\end{pmatrix}2\begin{pmatrix}3&4\\5&6\end{pmatrix}$$

$$=-8\begin{pmatrix}8&10\\8&10\end{pmatrix}=-16\begin{pmatrix}4&5\\4&5\end{pmatrix}$$

(ii) *How do we determine* $\mathbf{A}^2-\mathbf{B}^2$ *?*

$$\mathbf{A}^2-\mathbf{B}^2=\begin{pmatrix}1&2\\3&4\end{pmatrix}^2-\begin{pmatrix}5&6\\7&8\end{pmatrix}^2$$

$$=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}-\begin{pmatrix}5&6\\7&8\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}$$

$$=\begin{pmatrix}7&10\\15&22\end{pmatrix}-\begin{pmatrix}67&78\\91&106\end{pmatrix}$$

$$=\begin{pmatrix}-60&-68\\-76&-84\end{pmatrix}=-4\begin{pmatrix}15&17\\19&21\end{pmatrix}$$

2. We are given the matrix $\mathbf{A}=\frac{1}{7}\begin{pmatrix}3&-2&-6\\-2&6&-3\\-6&-3&-2\end{pmatrix}$.

(a) The matrix $\mathbf{A}^2$ is evaluated by

$$\mathbf{A}^2=\frac{1}{7}\begin{pmatrix}3&-2&-6\\-2&6&-3\\-6&-3&-2\end{pmatrix}\frac{1}{7}\begin{pmatrix}3&-2&-6\\-2&6&-3\\-6&-3&-2\end{pmatrix}$$

$$=\frac{1}{7}\times\frac{1}{7}\begin{pmatrix}3&-2&-6\\-2&6&-3\\-6&-3&-2\end{pmatrix}\begin{pmatrix}3&-2&-6\\-2&6&-3\\-6&-3&-2\end{pmatrix}$$

$$=\frac{1}{49}\begin{pmatrix}49&0&0\\0&49&0\\0&0&49\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}$$

Hence $\mathbf{A}^2=\mathbf{I}$. *What is* $\mathbf{A}^3$ *equal to?*

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \mathbf{IA} = \mathbf{A} = \frac{1}{7}\begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$$

(b) Since we have $\mathbf{A}^2 = \mathbf{I}$ therefore $\mathbf{A}^{-1} = \mathbf{A}$ and

$$\mathbf{A}^{2004} = \left(\mathbf{A}^2\right)^{1002} = \mathbf{I}^{1002} = \mathbf{I}$$

where **I** is the identity 3 by 3 matrix.

3. We have the following:

$$\mathbf{A}^T = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

The matrix multiplication **AB** is **not** valid because the number of columns (3) of matrix **A** does **not** equal the number of rows (4) of matrix **B**.
The matrix addition $\mathbf{B} + \mathbf{C}$ is **not** valid because matrices **B** and **C** are different sizes.
The matrix subtraction $\mathbf{A} - \mathbf{B}$ is **not** valid because matrices **A** and **B** are different sizes.
The matrix multiplication **CB** is given by

$$\mathbf{CB} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$$

The matrix multiplication $\mathbf{BC}^t$ is **not** valid because

$$\mathbf{BC}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 1 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}$$

This matrix multiplication is impossible because the number of columns (2) of the matrix **B** does **not** equal the number of rows (4) of the matrix $\mathbf{C}^T$.
The matrix multiplication $\mathbf{A}^2$ is given by

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -4 \\ 3 & 3 & 3 \\ 6 & 8 & 10 \end{pmatrix}$$

4. We are given the matrices $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$

$$\mathbf{BA}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}3 & 4\\ -1 & -2\end{pmatrix}$$

(b) We are given $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$:

$$\mathbf{AB}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}e & f\\ g & h\end{pmatrix}=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}$$

$$\mathbf{BA}=\begin{pmatrix}e & f\\ g & h\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}ae+cf & be+df\\ ag+ch & bg+dh\end{pmatrix}$$

Subtracting each corresponding entry gives

$$\mathbf{AB}-\mathbf{BA}=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}-\begin{pmatrix}ae+cf & be+df\\ ag+ch & bg+dh\end{pmatrix}$$

$$=\begin{pmatrix}bg-cf & af+bh-be-df\\ ce+dg-ag-ch & cf-bg\end{pmatrix}$$

5. We are given $\mathbf{A}=\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}$ which we can rewrite as $\mathbf{A}=\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$.

(i) We have

$$\mathbf{A}^2=\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$=\frac{1}{3^2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\frac{1}{3^2}\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$$

$$=\frac{2}{3^2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

(ii) Similarly we have

$$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}=\frac{1}{3^2}\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$=\frac{1}{3^3}\begin{pmatrix}4 & 4\\ 4 & 4\end{pmatrix}=\frac{2^2}{3^3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

6. Let $\mathbf{A}=\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$ then we have

$$\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix},\quad \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\\ 2\end{bmatrix},\quad \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$$

Expanding each of these out

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \text{ gives } \begin{matrix} b+c=-1 \\ e+f=0 \\ h+i=2 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \text{ gives } \begin{matrix} a+c=0 \\ d+f=-1 \\ g+i=2 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ gives } \begin{matrix} a+b=1 \\ d+e=1 \\ g+h=2 \end{matrix}$$

Solving these gives

$$a=1,\ b=0,\ c=-1,\ d=0,\ e=1,\ f=-1,\ g=1,\ h=1 \text{ and } i=1$$

Putting these values into the matrix **A** and checking:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix},\quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Thus $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

7. We have

$$\det(\mathbf{A}) = \det\begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$= 0 - 1\det\begin{bmatrix} 3 & 9 \\ 2 & 1 \end{bmatrix} + 5\det\begin{bmatrix} 3 & -6 \\ 2 & 6 \end{bmatrix}$$

$$= 0-(3-18)+5(18+12) = 15+150 = 165$$

Thus $\det(\mathbf{A}) = 165$.

8. First we find the determinant of the given matrix:

$$\det(\mathbf{A}) = \det\begin{bmatrix} 2 & -1 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + 1\det\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix} - 4\det\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= 2(3-2)+(-3+2)-4(-1+1) = 1$$

Since $\det(\mathbf{A}) = 1 \neq 0$ therefore the inverse of the matrix **A** exists.

Find the cofactors of each entry and adjoint which is the matrix of cofactors transposed . You can do this in your own time and establish that

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

9. The adjoint of the matrix is the matrix of cofactors transposed. Thus for the 2 by 2 matrix we have

$$adj\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\det \mathbf{A} = \det \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1 - 1 = 0$$

Since det**A** is zero therefore the inverse matrix does **not** exist.
For the 3 by 3 matrix we need to find the cofactors of each entry in order to determine *adj***A**.

Considering each entry starting from the top row of $\mathbf{A} = \begin{bmatrix} 3 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -2 & 5 \end{bmatrix}$ we have:

Cofactor of 3 is $\det \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = -5 + 6 = 1$.

Cofactor of -4 is $-\det \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -(5 - 6) = 1$.

Cofactor of 1 (in the top row) is $\det \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = (-2 + 2) = 0$.

Cofactor of 1 (in the middle row) is $-\det \begin{bmatrix} -4 & 1 \\ -2 & 5 \end{bmatrix} = -(-20 + 2) = 18$.

Cofactor of -1 is $\det \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = (15 - 2) = 13$.

Cofactor of 3 is $-\det \begin{bmatrix} 3 & -4 \\ 2 & -2 \end{bmatrix} = -(-6 + 8) = -2$.

Cofactor of 2 is $\det \begin{bmatrix} -4 & 1 \\ -1 & 3 \end{bmatrix} = (-12 + 1) = -11$.

Cofactor of -2 is $-\det \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = -(9 - 1) = -8$.

Cofactor of 5 is $\det \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = (-3 + 4) = 1$.

Thus the cofactor matrix **C** is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 18 & 13 & -2 \\ -11 & -8 & 1 \end{pmatrix}$$

The determinant of the matrix can be found by expanding along the top row, that is

$$\det \mathbf{A} = 3(\text{cofactor of } 3) - 4(\text{Cofactor of } -4) + 1(\text{Cofactor of } 1)$$
$$= 3(1) - 4(1) + 1(0) = -1$$

What is the adjoint matrix equal to?
It is the cofactor matrix transposed, that is

$$adj\mathbf{A} = \mathbf{C}^T = \begin{pmatrix} 1 & 1 & 0 \\ 18 & 13 & -2 \\ -11 & -8 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 18 & -11 \\ 1 & 13 & -8 \\ 0 & -2 & 1 \end{pmatrix}$$

Thus det**A=**-1. The inverse of the given matrix is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} adj\mathbf{A} = -1\begin{pmatrix} 1 & 18 & -11 \\ 1 & 13 & -8 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -18 & 11 \\ -1 & -13 & 8 \\ 0 & 2 & -1 \end{pmatrix}$$

10. (a) *What are the eigenvalues of* $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$ *equal to?*

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

(b) Let **u** and **v** be the corresponding eigenvectors of $\lambda_1 = 1$ and $\lambda_2 = 3$ respectively. We can find these by

$$(\mathbf{A} - \mathbf{I})\mathbf{u} = \begin{pmatrix} 1-1 & 1 \\ 0 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $x = 1$ and $y = 0$. Thus $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigenvector for $\lambda_1 = 1$. Similarly for $\lambda_2 = 3$ we have

$$(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \begin{pmatrix} 1-3 & 1 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $x = 1$ and $y = 2$. Thus $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *What is the matrix **Q** equal to?*

$\mathbf{Q} = (\mathbf{u} \quad \mathbf{v}) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. *What is the diagonal matrix **D** equal to?*

D is the diagonal matrix with entries along the leading diagonal given by the eigenvalues.
We have $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(c) *How do we find* $\mathbf{A}^5$ *?*

$\mathbf{A}^m = \mathbf{Q}\mathbf{D}^m\mathbf{Q}^{-1}$. We need to find the inverse of $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$:

$$\mathbf{Q}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

Substituting $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{Q}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ into $\mathbf{A}^m = \mathbf{Q}\mathbf{D}^m\mathbf{Q}^{-1}$ with $m = 5$ gives

$$\mathbf{A}^5 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^5 \frac{1}{2} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1^5 & 0 \\ 0 & 3^5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 243 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 1 & 243 \\ 0 & 486 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 2 & 242 \\ 0 & 486 \end{pmatrix} = \begin{pmatrix} 1 & 121 \\ 0 & 243 \end{pmatrix}$$

Hence $\mathbf{A}^5 = \begin{pmatrix} 1 & 121 \\ 0 & 243 \end{pmatrix}$.

11. We have

$$\mathbf{Au} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$
$$\mathbf{Av} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$
$$\mathbf{Aw} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence the following are the eigenvalues and corresponding eigenvectors of **A**:

$$\lambda_1 = 0,\ \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},\ \lambda_2 = 0,\ \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \lambda_3 = 3,\ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

12. (a) The eigenvalues λ of $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ are given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$
$$= (6-\lambda)(3-\lambda) - 4$$
$$= (\lambda-6)(\lambda-3) - 4$$
$$= \lambda^2 - 9\lambda + 14 = (\lambda-2)(\lambda-7) = 0$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 7$. Let **u** be the eigenvector belonging to $\lambda_1 = 2$.

$$(\mathbf{A}-2\mathbf{I})\mathbf{u}=\begin{bmatrix}6-2 & 2\\ 2 & 3-2\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=1 \text{ and } y=-2$$

Let **v** be the eigenvector belonging to $\lambda_2=7$:

$$(\mathbf{A}-7\mathbf{I})\mathbf{v}=\begin{bmatrix}6-7 & 2\\ 2 & 3-7\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{bmatrix}-1 & 2\\ 2 & -4\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=2 \text{ and } y=1$$

We have the eigenvalues and eigenvectors given by

$$\lambda_1=2,\ \mathbf{u}=\begin{pmatrix}1\\ -2\end{pmatrix} \text{ and } \lambda_2=7,\ \mathbf{v}=\begin{pmatrix}2\\ 1\end{pmatrix}$$

Note that since the given matrix **A** is symmetric therefore the eigenvectors **u** and **v** are orthogonal.

(b) Let $\mathbf{A}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$ then

$$\mathbf{A}\vec{v}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\\ -1\end{bmatrix}=\begin{bmatrix}2\\ 2\\ 2\\ -2\end{bmatrix}=2\begin{bmatrix}1\\ 1\\ 1\\ -1\end{bmatrix}=2\vec{v}$$

$$\mathbf{A}\vec{w}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\\ -1\\ 2\end{bmatrix}=\begin{bmatrix}1\\ -1\\ -1\\ 5\end{bmatrix}\neq\lambda\begin{bmatrix}1\\ -1\\ -1\\ 2\end{bmatrix}$$

Hence $\vec{v}$ is an eigenvector belonging to an eigenvalue of 2 but $\vec{w}$ is **not** an eigenvector of **A**.

13. The eigenvalues and eigenvectors for the matrix $\mathbf{A}=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$ can be evaluated by the usual procedure outlined above. We obtain the following:

$$\lambda_1=1,\ \mathbf{u}=\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix},\quad \lambda_2=2,\ \mathbf{v}=\begin{pmatrix}0\\ 1\\ 1\end{pmatrix} \text{ and } \lambda_3=0,\ \mathbf{w}=\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$$

(b) The given matrix **A** is diagonalizable because

$$\mathbf{P}=(\mathbf{u}\quad \mathbf{v}\quad \mathbf{w})=\begin{pmatrix}-1 & 0 & 1\\ 1 & 1 & -1\\ 1 & 1 & 0\end{pmatrix}$$

Therefore $\det(\mathbf{P}) = \det\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} = -1$. Since $\det(\mathbf{P}) \neq 0$ therefore we can diagonalise matrix **A**.