Complete Solutions to Exercise 16(g)

1. (i) The mean $« = E(T)$ and the formula is given by

$$« = E(T) = \int_0^{10} t\left(\frac{t}{50}\right) dt = \frac{1}{50}\int_0^{10}\left(t^2\right) dt \qquad \left[\text{Taking out } \frac{1}{50}\right]$$

$$= \frac{1}{50}\left[\frac{t^3}{3}\right]_0^{10} \qquad [\text{Integrating}]$$

$$= \frac{1}{150}\left[10^3 - 0^3\right] = 6.67$$

(ii) $E\left(T^2\right)$ can be evaluated by using the formula $E\left(T^2\right) = \int_0^{\infty} t^2 f(t)\, dt$. We are given that $f(t) = \frac{t}{50}$ for t between 0 and 10 therefore

$$E\left(T^2\right) = \int_0^{10} t^2\left(\frac{t}{50}\right) dt = \frac{1}{50}\int_0^{10}\left(t^3\right) dt$$

$$= \frac{1}{50}\left[\frac{t^4}{4}\right]_0^{10} = \frac{1}{200}\left[10^4 - 0^4\right] = 50$$

(iii) The variance $±^2$ is given by the formula $±^2 = E\left(T^2\right) - «^2$. Substituting the results of parts (i) and (ii), $« = 6.67$ and $E\left(T^2\right) = 50$, into this formula:

$$±^2 = E\left(T^2\right) - «^2 = 50 - 6.67^2 = 5.5111$$

(iv) The probability that the waiting time is at most 5 minutes is given by

$$P(0 \le T \le 5) = \int_0^5 \left(\frac{t}{50}\right) dt = \frac{1}{50}\int_0^5 t\, dt$$

$$= \frac{1}{50}\left[\frac{t^2}{2}\right]_0^5 = \frac{1}{100}\left[5^2 - 0^2\right] = \frac{25}{100} = \frac{1}{4}$$

(v) Using the result of part (iv) we have that the probability of the waiting time is more than 5 minutes is $1 - \frac{1}{4} = \frac{3}{4}$.

2. (a) The probability that the television breaks down within a year is given by

$$P(0 \le T \le 1) = \int_0^1 \frac{t^3}{2^{10}}\, dt$$

$$= \frac{1}{2^{10}}\int_0^1 t^3\, dt = \frac{1}{2^{10}}\left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4\times 2^{10}}\left[1^4 - 0^4\right] = \frac{1}{4096}$$

(b) Similarly the probability between the 1st and 2nd year is

$$P(1 \le T \le 2) = \int_1^2 \frac{t^3}{2^{10}}\, dt$$

$$= \frac{1}{2^{10}}\int_1^2 t^3\, dt = \frac{1}{2^{10}}\left[\frac{t^4}{4}\right]_1^2 = \frac{1}{4\times 2^{10}}\left[2^4 - 1^4\right] = \frac{15}{4096}$$

(c) The probability of the television breaking down after 6 years is $P(6 \le T \le 8)$ because the television does **not** last more than 8 years. Thus we have

$$P(6 \le T \le 8) = \int_6^8 \frac{t^3}{2^{10}}\, dt$$

$$= \frac{1}{2^{10}} \int_6^8 t^3\, dt = \frac{1}{2^{10}} \left[\frac{t^4}{4} \right]_6^8 = \frac{1}{4 \times 2^{10}} \left[8^4 - 6^4 \right] = \frac{2800}{4096} = \frac{175}{256}$$

(d) The probability that the television breaks down within 2 years or after 6 years is given by adding the answers to parts (a), (b) and (c). We have

$$P(0 \le T \le 2 \text{ or } 6 \le T \le 8) = P(0 \le T \le 2) + P(6 \le T \le 8)$$

$$= P(0 \le T \le 1) + P(1 \le T \le 2) + P(6 \le T \le 8)$$

$$= \frac{1}{4096} + \frac{15}{4096} + \frac{2800}{4096} = \frac{2816}{4096} = \frac{11}{16}$$

3. (a) The mean time for completion « $= E(T) = \int_0^\infty t f(t)\, dt$ with $f(t) = \frac{t}{48}\left(1 - \frac{t}{24}\right)$:

« $= E(T) = \int_0^{12} t \frac{t}{48}\left(1 - \frac{t}{24}\right) dt$

$$= \frac{1}{48} \int_0^{12} t^2 \left(\frac{24 - t}{24} \right) dt$$

$$= \frac{1}{48 \times 24} \int_0^{12} \left(24t^2 - t^3 \right) dt$$

$$= \frac{1}{1152} \left[8t^3 - \frac{t^4}{4} \right]_0^{12} = \frac{1}{1152} \left[8(12)^3 - \frac{12^4}{4} \right] = \frac{8640}{1152} = 7\frac{1}{2}$$

The means time for completion is $7\frac{1}{2}$ hours.

(b) We evaluate $E(T^2) = \int_0^\infty t^2 f(t)\, dt$ with $f(t) = \frac{t}{48}\left(1 - \frac{t}{24}\right)$ and limits 0, 12:

$$E(T^2) = \int_0^{12} t^2 \frac{t}{48}\left(1 - \frac{t}{24}\right) dt$$

$$= \frac{1}{48} \int_0^{12} t^3 \left(\frac{24 - t}{24} \right) dt$$

$$= \frac{1}{48 \times 24} \int_0^{12} \left(24t^3 - t^4 \right) dt$$

$$= \frac{1}{1152} \left[6t^4 - \frac{t^5}{5} \right]_0^{12} = \frac{1}{1152} \left[6(12)^4 - \frac{12^5}{5} \right] = 64\frac{4}{5}$$

(c) The variance is given by the formula ± $^2 = E(T^2) -$ « ^{2}. By substituting our answers from parts (a) and (b), « $= 7\frac{1}{2}$ and $E(T^2) = 64\frac{4}{5}$ into this formula we have

$$\pm^2 = E\left(T^2\right) - \text{«}^2 = 64\frac{4}{5} - \left(7\frac{1}{2}\right)^2 = 8\frac{11}{20}$$

13. (i) The mean $\text{«} = E(T)$ which means that

$$\text{«} = E(T) = \int_a^b t\left(\frac{1}{b-a}\right)dt$$

$$= \frac{1}{b-a}\int_a^b t\,dt \qquad \left[\text{Taking out } \frac{1}{b-a}\right]$$

$$= \frac{1}{b-a}\left[\frac{t^2}{2}\right]_a^b \qquad [\text{Integrating}]$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right] = \frac{1}{2(b-a)}\left[(b-a)(b+a)\right] = \frac{b+a}{2}$$

This is our required result.

(ii) To find the standard deviation we first determine the variance $\pm^2$ and then take the square root:

$$\pm^2 = E\left(T^2\right) - \text{«}^2$$

$$= E\left(T^2\right) - \left(\frac{a+b}{2}\right)^2 \qquad \left[\text{Because by part (i)} \quad \text{«} = \frac{a+b}{2}\right]$$

We need to determine $E\left(T^2\right)$.

$$E\left(T^2\right) = \int_a^b t^2\left(\frac{1}{b-a}\right)dt$$

$$= \frac{1}{b-a}\int_a^b \left(t^2\right)dt \qquad \left[\text{Taking out } \frac{1}{b-a}\right]$$

$$= \frac{1}{b-a}\left[\frac{t^3}{3}\right]_a^b \qquad [\text{Integrating}]$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right]$$

$$= \frac{1}{3(b-a)}\left[(b-a)\left(b^2 + ab + a^2\right)\right] = \frac{b^2 + ab + a^2}{3} \qquad [\text{Cancelling } b-a]$$

Substituting $E\left(T^2\right) = \frac{b^2 + ab + a^2}{3}$ into $\pm^2 = E\left(T^2\right) - \left(\frac{a+b}{2}\right)^2$ gives

$$\pm^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4} \qquad \begin{bmatrix}\text{Expanding brackets} \\ (a+b)^2 = a^2+2ab+b^2\end{bmatrix}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12} \qquad [\text{Common denominator}]$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

Taking the square root of this gives $\pm = \sqrt{\frac{(b-a)^2}{12}} = \frac{b-a}{\sqrt{12}}$.