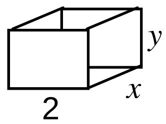
Solutions to Miscellaneous Exercise 7

1.



The volume $= 100 \text{ m}^3$. So

$$2xy = 100$$

$$y = \frac{100}{2x} = \frac{50}{x} \qquad (*)$$

The surface area, A, consists of the bottom part, 2 sides, the front and the back. Thus

$$A = \underbrace{2x}_{\text{bottom}} + \underbrace{(2y + 2y)}_{\text{front and back}} + \underbrace{(xy + xy)}_{\text{both sides}} = 2x + 4y + 2xy \underbrace{=}_{\substack{\text{substituting}\\ \text{from (*)}}} 2x + 4\left(\frac{50}{x}\right) + 2x\left(\frac{50}{x}\right)$$

$$A = 2x + \frac{200}{x} + 100 = 2x + 200x^{-1} + 100$$

For stationary points:

$$\frac{dA}{dx} = 2 - 200x^{-2} = 0$$

$$2 = 200x^{-2} = \frac{200}{x^2}$$

$$x^2 = \frac{200}{2} = 100$$

How can we find x*?*

Take the square root of both sides: $x = \sqrt{100} = +10, \ -10$

Since x is length it cannot be -10. So $x = 10\,\text{m}$.

To check that $x = 10\,\text{m}$ gives minimum surface area we have to differentiate again:

$$\frac{dA}{dx} = 2 - 200x^{-2}$$

$$\frac{d^2A}{dx^2} = 400x^{-3} = \frac{400}{x^3} > 0 \quad (\text{because } x > 0)$$

By (7.3), $x = 10\,\text{m}$ gives minimum surface area. *What is the value of* y*?*

We can find y from (*) by substituting $x = 10$: $y = 50/10 = 5\,\text{m}$.

Thus $x = 10\,\text{m}$, $y = 5\,\text{m}$ gives minimum surface area.

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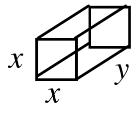
2. Let x and y represent the dimensions as shown below:



The perimeter of the cross section is $4x$ so

$$4x + y = 2$$

$$y = 2 - 4x \qquad (*)$$

The volume, v, of the parcel is given by

$$v = x^2 y = x^2(2 - 4x) = 2x^2 - 4x^3$$

(7.3) $\qquad A' = 0, \ A'' > 0 \ \text{ minimum}$

To find stationary points

$$\frac{dv}{dx} = 4x - 12x^2 = 0,\ \ 4x(1-3x) = 0 \text{ which gives } x = 0,\ \ x = 1/3$$

$x = 0$ m is not a feasible solution, *why not?*

If $x = 0$ m then we will not have a parcel. For $x = 1/3$ m, we can use the second derivative test:

$$\frac{d^2v}{dx^2} = 4 - 24x$$

Substituting $x = \frac{1}{3}$ gives $\frac{d^2v}{dx^2} = 4 - 8 = -4 < 0$. By (7.2), $x = \frac{1}{3}$ m gives maximum volume. To find y we substitute $x = 1/3$ into (*):

$$y = 2 - \frac{4}{3} = \frac{2}{3}$$

Hence $x = 1/3$ m, $y = 2/3$ m gives maximum volume.

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3. Similar to solution 2. Let L represent the sum of length and girth.

$$4x + y = L$$

$$y = L - 4x \qquad\qquad (\dagger)$$

The volume v is given by

$$v = x^2 y = x^2 (L - 4x)$$

$$v = Lx^2 - 4x^3$$

$$\frac{dv}{dx} = 2Lx - 12x^2 = 0,\ \ 2x(L - 6x) = 0 \text{ gives } \ x = 0,\ \ x = L/6$$

As before $x = L/6$. Differentiating again:

$$\frac{d^2v}{dx^2} = 2L - 24x$$

At $x = \frac{L}{6}$, $\frac{d^2v}{dx^2} = 2L - 24\left(\frac{L}{6}\right) = 2L - 4L = -2L < 0$ [Negative]

By (7.2), $x = L/6$ gives maximum volume. Substituting $x = L/6$ into $(\dagger)$:

$y = L - \frac{4.L}{6} = \frac{2L}{6} = 2x$. Hence the length is twice the side of the square.

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4. We have

$$w = -36x^2 + 50x$$

$$\frac{dw}{dx} = -72x + 50 = 0,\ \ 72x = 50 \text{ gives } x = \frac{50}{72} = \frac{25}{36} \text{ m}$$

$$\frac{d^2w}{dx^2} = -72 < 0 \qquad\qquad [\text{Negative}]$$

By (7.2) at $x = 25/36$ m the loading is maximum .

5.

$$y = \frac{1}{12\times10^3}\left(x^4 - 14x^3 + 36x^2\right) \qquad\qquad (*)$$

$$\frac{dy}{dx} = \frac{1}{12\times10^3}\left(4x^3 - 42x^2 + 72x\right)$$

(7.2) $\qquad y' = 0,\ \ y'' < 0$ maximum

For stationary points we have

$$\frac{2x}{12\times10^3}\left(2x^2-21x+36\right)=0$$

$$x=0 \quad \text{or} \quad 2x^2-21x+36=0$$

We solve the quadratic by putting $a=2,\ b=-21$ and $c=36$ into (1.16), which gives:

$$x=\frac{21\pm\sqrt{(-21)^2-(4\times2\times36)}}{4}=2.16\text{ m or }8.34\text{ m}$$

x cannot be 8.34 m because the beam is only 3m long so $x=2.16\text{ m}$. To check the nature of the stationary point we need to differentiate again:

$$\frac{dy}{dx}=\frac{1}{12\times10^3}\left(4x^3-42x^2+72x\right)$$

$$\frac{d^2y}{dx^2}=\frac{1}{12\times10^3}\left(12x^2-84x+72\right)\underbrace{=}_{\substack{\text{taking out a}\\ \text{factor of 12}}}\frac{12}{12\times10^3}\left(x^2-7x+6\right)=\left(x^2-7x+6\right)\times10^{-3}$$

At $x=2.16$, $\dfrac{d^2y}{dx^2}=\left(2.16^2-(7\times2.16)+6\right)\times10^{-3}=-4.45\times10^{-3}<0$ [Negative]

By (7.2), $x=2.16\text{ m}$ gives maximum deflection. To find the maximum deflection we substitute $x=2.16$ into (*):

$$y=\frac{1}{12\times10^3}\left[2.16^4-\left(14\times2.16^3\right)+\left(36\times2.16^2\right)\right]=4.05\times10^{-3}\text{ m}$$

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6. We have $x=2.5\sin(2\theta)$. For stationary points:

$$\frac{dx}{d\theta}=5\cos(2\theta)=0$$

$$\cos(2\theta)=0,\ 2\theta=\cos^{-1}(0)=\frac{\pi}{2}\text{ gives }\theta=\frac{\pi}{4}$$

Using the second derivative test:

$$\frac{d^2x}{d\theta^2}=-10\sin(2\theta)\qquad\left[\text{By }\frac{d}{d\theta}\left[\cos(k\theta)\right]=-k\sin(k\theta)\right]$$

At $\theta=\dfrac{\pi}{4}$, $\dfrac{d^2x}{d\theta^2}=-10\sin\left(2\times\dfrac{\pi}{4}\right)=-10<0.$

By (7.2), when $\theta=\pi/4$ the horizontal distance x is a maximum.

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7. Similar to solution 6. We can rewrite x as:

$$x=\frac{u^2}{2\times25}\left[2\sin(\theta)\cos(\theta)\right]=\frac{u^2}{50}\underbrace{\sin(2\theta)}_{\text{by (4.53)}}$$

Differentiating with respect to θ gives:

$$\frac{dx}{d\theta}=\frac{u^2}{50}2\cos(2\theta)=\frac{u^2}{25}\cos(2\theta)$$

By solution 6 we have a stationary point at $\theta=\pi/4$.

(1.16) $\quad x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

(4.53) $\quad 2\sin(x)\cos(x)=\sin(2x)$

(7.2) $\quad y'=0,\ y''<0$ maximum

To show that $\theta = \pi/4$ gives maximum:

$$\frac{dx}{d\theta} = \frac{u^2}{25}\cos(2\theta)$$

$$\frac{d^2x}{d\theta^2} = \frac{u^2}{25}\left[-2\sin(2\theta)\right] \qquad \left(\text{By } \frac{d}{d\theta}\left(\cos(k\theta)\right) = -k\sin(k\theta)\right)$$

At $\theta = \frac{\pi}{4}$, $\frac{d^2x}{d\theta^2} = -\frac{2u^2}{25}\underbrace{\sin\left(2\times\frac{\pi}{4}\right)}_{=1} = -\frac{2u^2}{25} < 0$ [Negative]

By (7.2), $\theta = \pi/4$ gives maximum x.

8. We have $s = 2 - te^{-t}$. The velocity, v, is found by differentiating:

$$v = \frac{ds}{dt} = 0 - \underbrace{\left[e^{-t} + t\left(-e^{-t}\right)\right]}_{\text{by (6.31)}} = -e^{-t}(1-t)$$

$$v = (t-1)e^{-t}$$

We need to differentiate v with respect to t to find the acceleration, a.

$$a = \frac{dv}{dt} \underbrace{=}_{\text{by (6.31)}} -e^{-t}(t-1) + e^{-t}(1)$$

$$= e^{-t}\left[-t+1+1\right]$$

$$= e^{-t}(2-t)$$

The graph $v = e^{-t}(t-1)$ cuts the v axis at $t = 0$, therefore substituting $t = 0$

$$v = e^0(0-1) = -e^0 = -1$$

Also $v = e^{-t}(t-1)$ cuts the t axis at $v = 0$,

$$e^{-t}(t-1) = 0,\ t-1 = 0 \text{ gives } t = 1$$

The graph v goes through $(0,-1)$ and $(1,0)$. *What happens to* $v = e^{-t}(t-1)$ *as* $t \to \infty$*?*

As $t \to \infty$, $v \to 0$ because e^{-t} is decaying as t increases.

What else can we discover about the graph?

Any stationary points and their nature.

$$v = e^{-t}(t-1)$$

$$\frac{dv}{dt} = a = (2-t)e^{-t} = 0 \text{ gives } t = 2$$

There is a stationary point at $t = 2$. To identify the nature of stationary point we differentiate again

$$\frac{d^2v}{dt^2} \underbrace{=}_{\text{by (6.31)}} (-1)e^{-t} + (2-t)\left(-e^{-t}\right) = (t-3)e^{-t}$$

At $t = 2$, $\frac{d^2v}{dt^2} = (2-3)e^{-2} = -e^{-2} < 0$. By (7.2), at $t = 2$, v has a maximum. The maximum value$= e^{-2}(2-1) = e^{-2}$. Also $\frac{d^2v}{dt^2} = 0$ when $t = 3$. Hence there is a general point of inflexion at $t = 3$ when $v = 2e^{-3}$. We have

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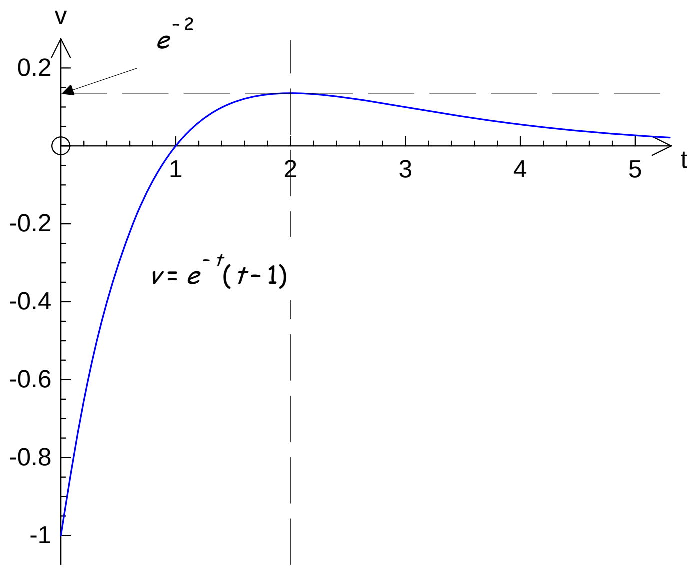
(6.31) $\qquad (uv)' = u'v + uv'$



9. From chapter 5 we know the exponential function is never zero, so $v = 4e^{-50t^2} \neq 0$ for any values of t. However as $t \to \pm\infty$, $v \to 0$ because the exponential function, e^{-50t^2}, decays as $t \to \pm\infty$. We can find the stationary points: $v = 4e^{-50t^2}$

$$\frac{dv}{dt} = 4e^{-50t^2}(-100t) = -400te^{-50t^2} = 0 \text{ gives } t = 0$$

Substituting $t = 0$, $v = 4e^0 = 4$. Hence $(0,4)$ is the stationary point of $v = 4e^{-50t^2}$. *What about the nature of the stationary point?*

We can use first derivative test: $\dfrac{dv}{dt} = -400te^{-50t^2}$

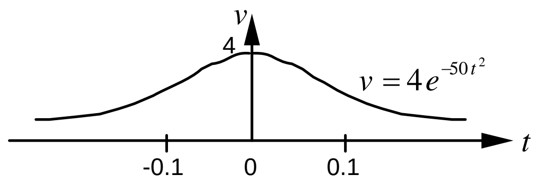
If $t < 0$ then $\dfrac{dv}{dt} > 0$ because the exponential part e^{-50t^2} is positive and we have -400 multiplied by another negative, t, which gives a positive answer.

If $t > 0$ then $\dfrac{dv}{dt} < 0$. By (7.7) the stationary point $(0,4)$ is a maximum of v.

To find general points of inflexion, we must differentiate again:

$$\frac{d^2v}{dt^2} = -400\left(\frac{d}{dt}\left[te^{-50t^2}\right]\right)$$
$$= -400\left[e^{-50t^2} - 100t^2e^{-50t^2}\right]$$

For inflexion, $\dfrac{d^2v}{dt^2} = -400e^{-50t^2}\left[1-100t^2\right] = 0$ gives $t^2 = \dfrac{1}{100}$, $t = \pm\dfrac{1}{10}$. Hence



10. We have

$$R = \frac{\ln(t/t_1)}{2\pi k} + \frac{1}{2\pi th} = \frac{1}{2\pi}\left[\frac{1}{k}\ln\left(\frac{t}{t_1}\right) + \frac{t^{-1}}{h}\right] \qquad \text{(Factorizing)}$$

For stationary points we need to differentiate R with respect to t:

$$\frac{dR}{dt}=\frac{1}{2\pi}\left[\frac{1}{k}.\frac{1}{t/t_1}.\left(\frac{1}{t_1}\right)-\frac{t^{-2}}{h}\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{k}.\frac{1}{t}-\frac{1}{t^2h}\right] \qquad \text{(Cancelling } t\text{'s)}$$

For stationary points we need $\frac{dR}{dt}=0$, thus

$$\frac{1}{kt}-\frac{1}{t^2h}=0 \qquad \left(\text{because } \frac{1}{2\pi} \text{ cannot be zero}\right)$$

$$\frac{1}{kt}=\frac{1}{t^2h} \text{ gives } t=\frac{k}{h} \text{ [Transposing]}$$

So thickness $t=k/h$ gives a stationary point. *How do we show this value gives minimum R?* Use the second derivative test:

$$\frac{dR}{dt}=\frac{1}{2\pi}\left[\frac{1}{kt}-\frac{1}{t^2h}\right]=\frac{1}{2\pi}\left[\frac{t^{-1}}{k}-\frac{t^{-2}}{h}\right]$$

$$\frac{d^2R}{dt^2}=\frac{1}{2\pi}\left[\frac{-t^{-2}}{k}+\frac{2t^{-3}}{h}\right]=\frac{1}{2\pi}\left[-\frac{1}{kt^2}+\frac{2}{t^3h}\right]$$

Substituting $t=k/h$:

$$\frac{d^2R}{dt^2}=\frac{1}{2\pi}\left[-\frac{1}{k\left(k/h\right)^2}+\frac{2}{\left(k/h\right)^3h}\right]$$

$$=\frac{1}{2\pi}\left[-\frac{h^2}{k^3}+\frac{2h^2}{k^3}\right]$$

$$=\frac{1}{2\pi}\left[\frac{h^2}{k^3}\right]>0 \qquad \left(\text{since } k>0\right)$$

Hence by (7.3), thickness $t=k/h$ gives minimum resistance R.

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11. We have $\alpha=\frac{n^2+12}{3-n}$. *How do we differentiate this?*

You can apply long division to rewrite α or use the quotient rule (6.32):

$$u=n^2+12 \qquad v=3-n$$

$$u'=2n \qquad v'=-1$$

$$\frac{d\alpha}{dn}=\frac{u'v-uv'}{v^2}$$

$$=\frac{2n\left(3-n\right)+\left(n^2+12\right)}{\left(3-n\right)^2}$$

$$=\frac{6n-2n^2+n^2+12}{\left(3-n\right)^2}$$

$$=\frac{12+6n-n^2}{\left(3-n\right)^2}$$

(6.32) $\quad \left(u/v\right)'=\left(u'v-uv'\right)/v^2$

(7.3) $\quad R'=0,\ R''>0$ minimum

For $\dfrac{d\alpha}{dn}=0$, $\quad 12+6n-n^2=0$ [Numerator=0]

Multiplying by -1 gives the quadratic $n^2-6n-12=0$ *How do we solve this?*
Substituting $a=1$, $b=-6$ and $c=-12$ into the quadratic formula:

$$n=\frac{6\pm\sqrt{36+(4\times 12)}}{2}$$
$$=7.58 \text{ or } -1.58$$

Hence $n=7.58$ (cannot have a negative gear ratio).
How can we show $n=7.58$ gives maximum acceleration, α?
Use the first derivative test:

$$\frac{d\alpha}{dn}=\frac{12+6n-n^2}{(3-n)^2}$$

We only need to examine the sign of the numerator because the denominator is positive.
If $n>7.58$, try $n=8$, then $12+(6\times 8)-8^2=-4<0$
If $n<7.58$, try $n=7$, then $12+(6\times 7)-7^2=5>0$
By (7.7), $n=7.58$ gives maximum acceleration.

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12. Replacing e^x with the Maclaurin series expansion of (7.15) we have:

$$\frac{e^x-1}{x}=\frac{\left(1+x+x^2/2!+x^3/3!+\ldots\right)-1}{x}$$
$$=\frac{x+x^2/2!+x^3/3!+\ldots}{x}$$
$$=\frac{x\left(1+x/2!+x^2/3!+\ldots\right)}{x}$$
$$=1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots$$

So $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\lim\limits_{x\to 0}\left(1+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\ldots\right)=1$

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13. The gradient, m, of the tangent is evaluated by differentiating $y=\sin^2(x)$:

$$\frac{dy}{dx}=2\sin(x)\cos(x)$$

At $x=\dfrac{\pi}{4}$, $\dfrac{dy}{dx}=2\sin\left(\dfrac{\pi}{4}\right)\cos\left(\dfrac{\pi}{4}\right)=1$. Hence $m=1$. Equation of tangent is of the form $y=x+c$. *How can we find c?*

At $x=\dfrac{\pi}{4}$, $y=\left[\sin\left(\dfrac{\pi}{4}\right)\right]^2=\left(\dfrac{1}{\sqrt{2}}\right)^2=\dfrac{1}{2}$, so the tangent goes through $x=\dfrac{\pi}{4}$, $y=\dfrac{1}{2}$.
Substituting these gives:

(7.15) $\qquad e^x=1+x+x^2/2!+x^3/3!+\ldots$

$$\frac{1}{2} = \frac{\pi}{4} + c$$

$$c = \frac{1}{2} - \frac{\pi}{4} = \frac{2}{4} - \frac{\pi}{4} = \frac{1}{4}(2 - \pi)$$

Therefore the equation of the tangent is $y = x + \frac{1}{4}(2 - \pi)$. *How do we find the equation of the normal?*

The gradient of the normal $= -1$ so the equation of the normal is of the form:

$$y = -x + c_1 \qquad (**)$$

The normal also goes through the point $x = \frac{\pi}{4}$, $y = \frac{1}{2}$. So

$$\frac{1}{2} = -\frac{\pi}{4} + c_1 \text{ gives } c_1 = \frac{1}{2} + \frac{\pi}{4} = \frac{2}{4} + \frac{\pi}{4} = \frac{1}{4}(2 + \pi)$$

Substituting $c_1 = \frac{1}{4}(2 + \pi)$ into (**) gives:

$$y = -x + \frac{1}{4}(2 + \pi) = \frac{1}{4}(2 + \pi) - x$$

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14. We need to differentiate $v = kx\ln\left(\frac{1}{x}\right)$, *how?*

First we can rewrite v as follows:

$$v = kx\ln\left(\frac{1}{x}\right)$$
$$= kx\ln\left(x^{-1}\right) = -kx\ln(x)$$

We can differentiate v by using the product rule, (6.31):

$$u = x \qquad w = \ln(x)$$
$$u' = 1 \qquad w' = 1/x$$

Applying (6.31)

$$\frac{dv}{dx} = -k\left[1.\ln(x) + x\left(\frac{1}{x}\right)\right] = -k\left[\ln(x) + 1\right]$$

For stationary points this is zero, therefore

$$-k\left[\ln(x) + 1\right] = 0$$
$$\ln(x) + 1 = 0 \qquad (\text{because } k > 0)$$
$$\ln(x) = -1$$

How can we find x from $\ln(x) = -1$?

Taking exponential of both sides gives $x = e^{-1}$.

Differentiate again to find whether this value, $x = e^{-1}$, gives maximum velocity.

$$\frac{dv}{dx} = -k\left[\ln(x) + 1\right]$$
$$\frac{d^2v}{dx^2} = -k\left(\frac{1}{x}\right) = -\frac{k}{x}$$

(6.31) $\qquad (uw)' = u'w + uw'$

Substituting $x = e^{-1}$ gives $\dfrac{d^2v}{dx^2} = -\dfrac{k}{e^{-1}} < 0$ because k and e^{-1} are both positive. By (7.2) the maximum velocity occurs at $x = e^{-1}$.

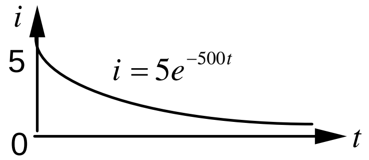
15. Substituting $i = 5e^{-500t}$ and $L = 2\times 10^{-3}$ into v gives

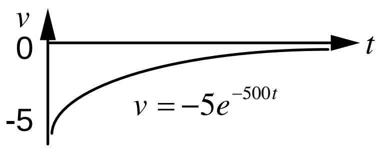
$$\begin{aligned} v &= \left(2\times 10^{-3}\right)\frac{d}{dt}\left(5e^{-500t}\right) \\ &= \left(2\times 10^{-3}\right)\left(-500\times 5e^{-500t}\right) \\ &= \left(2\times 10^{-3}\right)\left(-2500\right)e^{-500t} \\ &= -5e^{-500t} \end{aligned}$$

As $t\to\infty, i\to 0$ because exponential function, e^{-500t}, goes to zero. We also know it is a decaying graph because of the negative sign in front of the $500t$. What about stationary points:

$$i = 5e^{-500t},\ \frac{di}{dt} = -2500e^{-500t}$$

Putting this to zero gives $-2500e^{-500t} = 0$. *Where is this function zero?*
This function cannot be zero for any real values of t because it is the exponential function so there are no stationary points.
At $t = 0,\ i = 5e^0 = 5$. Thus we have:





16. Rewriting F we have:

$$F = \frac{Ir^2}{2}\left(x^2 + r^2\right)^{-3/2}$$

$$\frac{dF}{dx} = \frac{Ir^2}{2}\left(-\frac{3}{2}\right)\left(x^2 + r^2\right)^{-5/2}\left(2x\right) = -\frac{3Ir^2}{2}x\left(x^2 + r^2\right)^{-5/2}$$

$$\frac{dF}{dx} = -\frac{3Ir^2}{2}\frac{x}{\left(x^2 + r^2\right)^{5/2}}$$

Points of inflexion occurs at $\dfrac{d^2F}{dx^2} = 0$, so we need to differentiate again, *how?*

Use the quotient rule (6.32) with:

$$u = x \qquad v = \left(x^2 + r^2\right)^{5/2}$$

$$u' = 1 \qquad v' = \frac{5}{2}\left(x^2 + r^2\right)^{3/2}2x = 5x\left(x^2 + r^2\right)^{3/2}$$

(6.32) $\left(\dfrac{u}{v}\right)' = \dfrac{\left(u'v - uv'\right)}{v^2}$

(7.2) $v' = 0,\ v'' < 0$ maximum

$$\frac{d^2F}{dx^2} = -\frac{3Ir^2}{2}\left[\frac{\left(x^2+r^2\right)^{5/2} - 5x\left(x^2+r^2\right)^{3/2}x}{\left[\left(x^2+r^2\right)^{5/2}\right]^2}\right]$$

$$\frac{d^2F}{dx^2} = -\frac{3Ir^2}{2}\left[\frac{\left(x^2+r^2\right)^{5/2} - 5x^2\left(x^2+r^2\right)^{3/2}}{\left(x^2+r^2\right)^5}\right] \qquad (\dagger)$$

Putting this to zero gives that the numerator is zero:

$$-\frac{3Ir^2}{2}\left[\left(x^2+r^2\right)^{5/2} - 5x^2\left(x^2+r^2\right)^{3/2}\right] = 0$$

This can only occur if the terms inside the square brackets are zero because the current $I \neq 0$ and radius $r \neq 0$.

$$\left(x^2+r^2\right)^{5/2} - 5x^2\left(x^2+r^2\right)^{3/2} = 0$$

Factorizing: $$\left(x^2+r^2\right)^{3/2}\left[\left(x^2+r^2\right) - 5x^2\right] = 0$$

Again only the square brackets term can be zero because $\left(x^2+r^2\right)^{3/2} \neq 0$ (all terms are squared and no negative sign).

$$\left(x^2+r^2\right) - 5x^2 = 0 \text{ implies } r^2 - 4x^2 = 0 \text{ which gives } x = \pm\frac{r}{2}$$

Since x is distance, $x = \frac{r}{2}$. We need to check for change of sign of $\frac{d^2F}{dx^2}$. If $x < \frac{r}{2}$ then $r^2 - 4x^2 > 0$, hence $\frac{d^2F}{dx^2} < 0$ because there is a negative sign outside the square brackets in $(\dagger)$.

If $x > \frac{r}{2}$ then $r^2 - 4x^2 < 0$, hence $\frac{d^2F}{dx^2} > 0$. At $x = \frac{r}{2}$ we have a uniform field.

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17. *How can we differentiate* η *with respect to* x*?*

Use the quotient rule (6.32) with

$$u = xs\cos(\phi) \qquad v = L_i + xs\cos(\phi) + x^2L_c$$

$$u' = s\cos(\phi) \qquad v' = s\cos(\phi) + 2xL_c$$

Substituting these into (6.32) gives:

$$\frac{d\eta}{dx} = \frac{s\cos(\phi)\left(L_i + xs\cos(\phi) + x^2L_c\right) - xs\cos(\phi)\left(s\cos(\phi) + 2xL_c\right)}{\left(L_i + xs\cos(\phi) + x^2L_c\right)^2}$$

$$= \frac{L_is\cos(\phi) + xs^2\cos^2(\phi) + x^2L_cs\cos(\phi) - xs^2\cos^2(\phi) - 2x^2sL_c\cos(\phi)}{\left(L_i + xs\cos(\phi) + x^2L_c\right)^2}$$

$$= \frac{L_is\cos(\phi) - x^2sL_c\cos(\phi)}{\left(L_i + xs\cos(\phi) + x^2L_c\right)^2}$$

$$\frac{d\eta}{dx} = \frac{s\cos(\phi)\left(L_i - x^2L_c\right)}{\left(L_i + xs\cos(\phi) + x^2L_c\right)^2}$$

(6.32) $\quad (u/v)' = (u'v - uv')/v^2$

For stationary point $\frac{d\eta}{dx}=0$, hence the numerator $=0$. Since $s\cos(\phi)>0$ we have

$$L_i - x^2 L_c = 0$$

$$L_i = x^2 L_c,\ x^2 = \frac{L_i}{L_c} \text{ gives } x = \sqrt{\frac{L_i}{L_c}}$$

How can we show that this value of x gives maximum efficiency?
Use the first derivative test (7.7):

$$\frac{d\eta}{dx} = \frac{s\cos(\phi)\left(L_i - x^2 L_c\right)}{\left(L_i + xs\cos(\phi) + x^2 L_c\right)^2}$$

We only need to examine the term $L_i - x^2 L_c$ because the other terms are positive.

If $x < \sqrt{\frac{L_i}{L_c}}$ then $x^2 < \frac{L_i}{L_c}$ so $L_i - x^2 L_c > 0$ and $\frac{d\eta}{dx} > 0$

If $x > \sqrt{\frac{L_i}{L_c}}$ then $x^2 > \frac{L_i}{L_c}$ so $L_i - x^2 L_c < 0$ and $\frac{d\eta}{dx} < 0$

By (7.7), $x = \sqrt{\frac{L_i}{L_c}}$ gives maximum efficiency.

18.(i) Let $f(x) = \sinh(x)$ then

$$\begin{aligned}
f(x) &= \sinh(x) & f(0) &= \sinh(0) = 0\\
f'(x) &= \cosh(x) & f'(0) &= \cosh(0) = 1\\
f''(x) &= \sinh(x) & f''(0) &= 0\\
f'''(x) &= \cosh(x) & f'''(0) &= 1\\
f^{(4)}(x) &= \sinh(x) & f^{(4)}(0) &= 0\\
f^{(5)}(x) &= \cosh(x) & f^{(5)}(0) &= 1
\end{aligned}$$

Substituting these into (7.14) gives:

$$\begin{aligned}
\sinh(x) &= 0 + (1)x + 0 + (1)\frac{x^3}{3!} + 0 + (1)\frac{x^5}{5!} + \ldots\\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots
\end{aligned}$$

(ii) The MAPLE output is on the web site.

(7.14) $\quad f(x) = f(0) + f'(0)x + x^2 f''(0)/2! + x^3 f'''(0)/3! + \ldots$

19. (i) We let $f(x) = \tan^{-1}(x)$:

$$f(x) = \tan^{-1}(x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = -\frac{2x}{\left(1+x^2\right)^2} \qquad f''(0) = 0$$

$$f'''(x) = \frac{6x^2-2}{\left(1+x^2\right)^3} \qquad f'''(0) = -2$$

$$f^{(4)}(x) = \frac{24\left(x-x^3\right)}{\left(1+x^2\right)^4} \qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \frac{24\left(1-10x^2+5x^4\right)}{\left(1+x^2\right)^5} \qquad f^{(5)}(0) = 24$$

We have 3 non-zero terms; $f'(0)=1$, $f'''(0)=-2$ and $f^{(5)}(0)=24$.
Substituting these into (7.14) gives

$$\tan^{-1}(x) = 0 + (1\times x) + 0 + \left(-\frac{2}{3!}\right)x^3 + 0 + \left(\frac{24}{5!}\right)x^5 + \dots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} \dots \qquad (*)$$

(ii) To obtain the required result we need to substitute $x = 1$ into (*):

$$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} \dots$$

Remember $\tan^{-1}(1) = \frac{\pi}{4}$. Thus

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \dots$$

(iii) All evaluations equal $\pi/4$.

---

20. (i)

$$y = \frac{1}{3}x^3 - 3x^2 + 8x - 3$$

$$\frac{dy}{dx} = x^2 - 6x + 8$$

For turning point $\frac{dy}{dx} = 0$

$$x^2 - 6x + 8 = 0,\ (x-4)(x-2) = 0 \text{ gives } \quad x = 4 \text{ or } x = 2$$

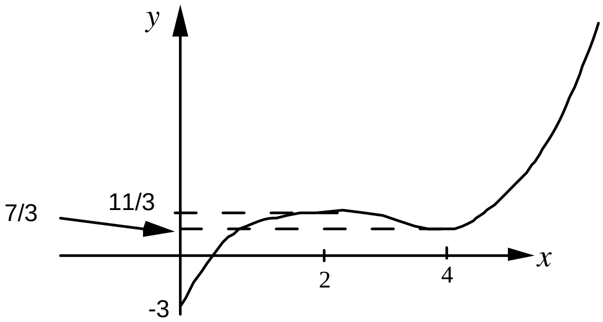
$$\frac{d^2y}{dx^2} = 2x - 6$$

At $x = 2$, $\frac{d^2y}{dx^2} = -2 < 0$ maximum, $y = \frac{11}{3}$

(7.14) $\qquad f(x) = f(0) + f'(0)x + x^2 f''(0)/2! + x^3 f'''(0)/3! + \dots$

At $x=4$, $\dfrac{d^2y}{dx^2}=2>0$ minimum, $y=\dfrac{7}{3}$

The curve $y=\dfrac{1}{3}x^3-3x^2+8x-3$ cuts the y axis at -3 (the value of y at $x=0$).



(ii) Let

$$f(x)=\frac{1}{3}x^3-3x^2+8x-3$$

$$f'(x)=x^2-6x+8$$

By looking at the graph, take $r_1=0$ (you could just as well take $r_1=1$)

$$r_2 \underbrace{=}_{\text{by (7.25)}} 0-\frac{f(0)}{f'(0)}=0.3750$$

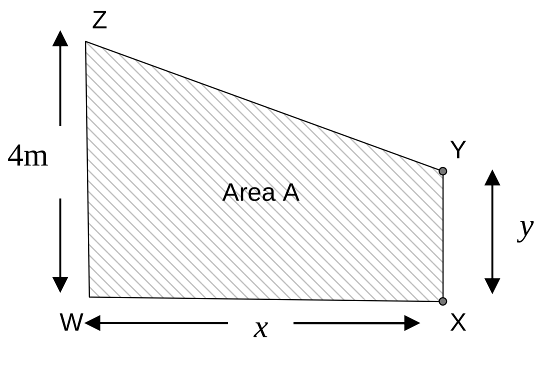
$$r_3=0.375-\frac{f(0.375)}{f'(0.375)}=0.4437$$

$$r_4=0.4437-\frac{f(0.4437)}{f'(0.4437)}=0.4458$$

$$r_5=0.4458-\frac{f(0.4458)}{f'(0.4458)}=0.4458$$

Since $r_4=r_5$, the root of $\dfrac{1}{3}x^3-3x^2+8x-3=0$ is 0.446 (3 d.p.).

21. (i) We use the trapezium rule to determine the area A in the given diagram



$$A=\frac{1}{2}x(4+y) \qquad (*)$$

We are given that

$$y+YZ=6 \text{ implies that } YZ=6-y$$

YZ can be found by Pythagoras:

$$YZ^2 = (6-y)^2 = (4-y)^2 + x^2$$

$$36 - 12y + y^2 = 16 - 8y + y^2 + x^2$$

Collecting like terms gives

$$20 - x^2 = 4y \text{ which gives } y = \frac{1}{4}\left(20 - x^2\right)$$

Substituting $y = \frac{1}{4}\left(20 - x^2\right)$ into (*) yields

$$A = \frac{1}{2}x\left(4 + \frac{1}{4}\left(20 - x^2\right)\right)$$

$$= \frac{1}{8}x\left(16 + 20 - x^2\right) = \frac{1}{8}x\left(36 - x^2\right) = \frac{1}{8}\left(36x - x^3\right)$$

(ii) For maximum cross-sectional area we differentiate the above function:

$$A = \frac{1}{8}\left(36x - x^3\right)$$

$$\frac{dA}{dx} = \frac{1}{8}\left(36 - 3x^2\right)$$

Stationary points occur where the derivative is zero:

$$\frac{1}{8}\left(36 - 3x^2\right) = 0 \quad \Rightarrow \quad 36 - 3x^2 = 0 \quad \Rightarrow \quad x^2 = 12 \quad \Rightarrow \quad x = \sqrt{12} = 2\sqrt{3}$$

To show that we have a maximum at this value of x we differentiate again:

$$\frac{dA}{dx} = \frac{1}{8}\left(36 - 3x^2\right)$$

$$\frac{d^2A}{dx^2} = \frac{1}{8}(0 - 6x)$$

Substituting $x = 2\sqrt{3}$ into $\frac{d^2A}{dx^2} = \frac{1}{8}(0 - 6x) = -\frac{6}{8}x$ gives a negative value so we have maximum at $x = 2\sqrt{3}$. We can substitute this value into $y = \frac{1}{4}\left(20 - x^2\right)$ to find y:

$$y = \frac{1}{4}\left(20 - x^2\right) = \frac{1}{4}\left(20 - \sqrt{12}^2\right) = 2$$

Hence $x = 2\sqrt{3}$ m and $y = 2$ m gives maximum cross-sectional area.

22. Using the binomial series, (7.24), with $x = -\frac{v}{c^2}$ we have

$$\left(1 - \frac{v^2}{c^2}\right)^{1/2} = 1 + \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \left[\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\right]\left(-\frac{v^2}{c^2}\right)^2 + \left[\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\right]\left(-\frac{v^2}{c^2}\right)^3 \dots$$

(7.24) $$(1+x)^n = 1 + nx + \left[\frac{n(n-1)}{2!}\right]x^2 + \left[\frac{n(n-1)(n-3)}{3!}\right]x^3 + \dots$$

(7.29) $$r_{n+1} = r_n + \frac{f(r_n)}{f'(r_n)}$$

$$=1-\frac{v^2}{2c^2}-\frac{v^4}{8c^4}-\frac{3v^6}{48c^6}-\ldots$$
$$=1-\frac{v^2}{2c^2}-\frac{v^4}{8c^4}-\frac{v^6}{16c^6}-\ldots$$

23. Similar to solution of question 22 but we ignore higher powers.
By using the binomial expansion we can show that

$$\frac{1}{\sqrt{1-x}}=(1-x)^{-1/2}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\cdots$$

Substituting $x=\left(\frac{v}{c}\right)^2$ because we are given $m=\frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ into the above:

$$m=\frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}=\frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}=m_0\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left[\left(\frac{v}{c}\right)^2\right]^2+\frac{5}{16}\left[\left(\frac{v}{c}\right)^2\right]^3+\cdots\right)$$
$$=m_0\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4+\frac{5}{16}\left(\frac{v}{c}\right)^6+\cdots\right)$$

We are told that v is very small compared to c therefore $\left(\frac{v}{c}\right)$ is a small number and taking powers makes it even smaller. Hence we ignore the higher powers of $\left(\frac{v}{c}\right)$, that is powers above 2. Hence we have

$$m=m_0\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2\right)=m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

Substituting this into the given formula for KE , $K=(m-m_0)c^2$, we have

$$K=\left(m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)-m_0\right)c^2$$
$$=\left(m_0+\frac{1}{2}\frac{v^2}{c^2}m_0-m_0\right)c^2=\left(\frac{1}{2}\frac{v^2}{c^2}m_0\right)c^2=\frac{1}{2}m_0v^2$$

This is our required result.