Complete Solutions to Miscellaneous Exercise 3

1. (i) *How do we find* $\left(f\left(\frac{1}{2}\right)\right)^2$?

We replace x with $1/2$ in $f(x)=\frac{2x-1}{1-x}$ and then we square our result:

$$\left(f\left(\frac{1}{2}\right)\right)^2 = \left(\frac{\left(2\times\frac{1}{2}\right)-1}{1-\frac{1}{2}}\right)^2 = 0^2 = 0$$

(ii) We first find $f(2)$ and $f(5)$ and then substitute these values into $\frac{6f(2)-3f(5)}{f(5)}$.

We have $f(2)=\frac{(2\times 2)-1}{1-2}=-3$, similarly $f(5)=\frac{10-1}{1-5}=-\frac{9}{4}$. Substituting $f(2)=-3$ and $f(5)=-\frac{9}{4}$ gives:

$$\frac{6f(2)-3f(5)}{f(5)}=\frac{[6\times(-3)]-[3\times(-9/4)]}{-9/4}=5$$

(iii) $f(f(x))=f\left(\frac{2x-1}{1-x}\right)=\frac{2\left(\frac{2x-1}{1-x}\right)-1}{1-\left(\frac{2x-1}{1-x}\right)}$. *How do we simplify this horrendous expression?*

Multiply numerator and denominator by $1-x$:

$$f(f(x))=\frac{2(2x-1)-(1-x)}{(1-x)-(2x-1)}=\frac{4x-2-1+x}{1-x-2x+1}$$

$$=\frac{5x-3}{2-3x}\quad(2-3x\neq 0)$$

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2. (i) Replace t with $t+h$: $f(t+h)=4.9(t+h)^2$

$$=4.9\underbrace{\left(t^2+2ht+h^2\right)}_{\text{by }(1.13)}$$

$$=4.9t^2+9.8ht+4.9h^2$$

(ii) $$\frac{f(t+h)-f(t)}{h}=\frac{\left(4.9t^2+9.8ht+4.9h^2\right)-4.9t^2}{h}$$

$$=\frac{9.8ht+4.9h^2}{h}$$

$$=\frac{\cancel{h}(9.8t+4.9h)}{\cancel{h}}\quad\text{(cancelling the } h\text{'s)}$$

$$=9.8t+4.9h$$

(iii) Substituting $t=1$ into the result of part (ii), $9.8t+4.9h$, gives $9.8+4.9h$.

(iv) $9.8+(4.9\times 10^{-20})\approx 9.8$ (substitute $t=1, h=10^{-20}$ into result of part (ii)).

(v) $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=\lim_{h\to 0}(9.8t+4.9h)=9.8t$, (since h is very close to zero).

(1.13) $$(a+b)^2=a^2+2ab+b^2$$

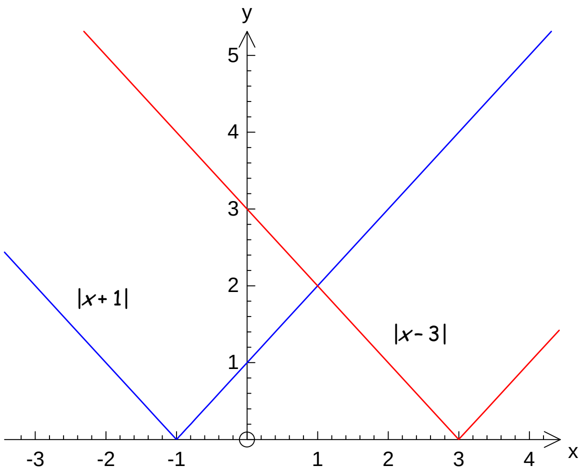
3. (i) Using the hint with $a = t$ and $b = h$ gives:

$$\frac{\phi(t+h)-\phi(t)}{h} = \frac{(t+h)^3 - t^3}{h} = \frac{(t^3 + 3t^2h + 3th^2 + h^3) - t^3}{h}$$
$$= \frac{3t^2h + 3th^2 + h^3}{h}$$
$$= \frac{\not{h}(3t^2 + 3th + h^2)}{\not{h}}$$
$$= 3t^2 + 3th + h^2$$

(ii) Since h is very close to zero we have

$$\lim_{h\to 0}\frac{\phi(t+h)-\phi(t)}{h} \underbrace{=}_{\text{by part (i)}} \lim_{h\to 0}(3t^2 + 3th + h^2) = 3t^2$$

4. The graphs of $|x+1|$ and $|x-3|$ are similar in shape to the graph of $|x|$ but shifted left and right respectively.



5. $f(-a) = 2(-a)^4 - (-a)^2 + 1001$
$= 2a^4 - a^2 + 1001$
$= f(a)$

6. $g(-a) = (-a)^5 - (-a)^3 - (-a)$
$= -a^5 + a^3 + a$
$= -(a^5 - a^3 - a)$
$= -g(a)$

7. Equating $f(x)$ and $g(x)$ gives $x^2 + 5 = 6x$. Subtracting $6x$ from both sides:

$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$
$$x = 5,\ x = 1$$

8. (i) $f(2) = 2^3 - (2\times 2^2) - 2 - 0.625 = -2.625$

(ii) $f(3) = 3^3 - (2\times 3^2) - 3 - 0.625 = 5.375$

Graph of f crosses the x- axis between $x = 2$ and $x = 3$ because f goes from being negative at $x = 2$ to positive at $x = 3$, therefore there is a root between 2 and 3. On MAPLE we plot f and then tune into the root.

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> f(x):=x^3-2*(x^2)-x-0.625;
> plot(f(x),x=2..3);
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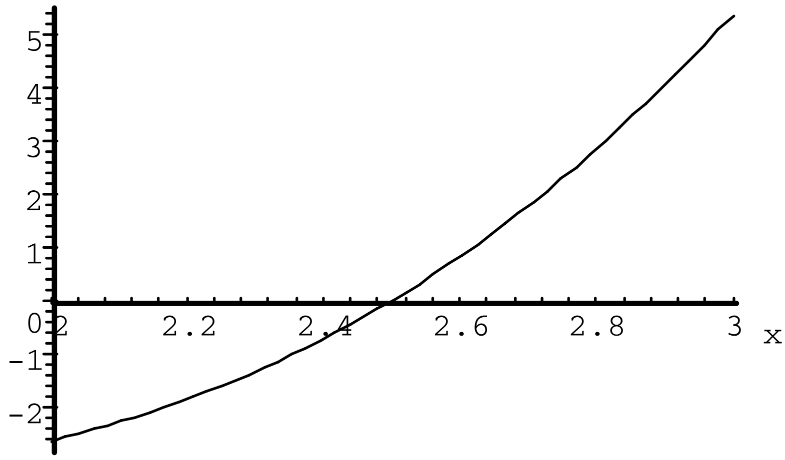

You can close in further by plotting the graph closer and closer to the point where the graph cuts the x-axis. Try plotting between 2.4 and 2.6. The actual root is 2.5

9. $f(x)=\frac{1}{x}$, $f^{-1}(x)=\frac{1}{x}$ (Self- Inverse). The graph of $\frac{1}{x}$ is shown in Fig 27 of Chapter 2.

10. *How do we find the inverse function?*

Let $y=\frac{ax-b}{cx-a}$ then extract x. Multiplying both sides by $cx-a$ gives:

$$(cx-a)y=ax-b$$
$$cxy-ay=ax-b$$
$$cxy-ax=ay-b$$
$$x(cy-a)=ay-b \text{ (factorizing the L .H.S.)}$$
$$x=\frac{ay-b}{cy-a}$$

Replacing the y with x gives the inverse function, thus

$$\phi^{-1}(x)=\frac{ax-b}{cx-a}$$

What do you notice about $\phi^{-1}(x)$?

$$\phi^{-1}(x)=\phi(x)$$

The function f has the same format as ϕ with $a=2,\ b=3$ and $c=5$. So what is $f^{-1}(x)$ equal to?

$$f^{-1}(x)=f(x)=\frac{2x-3}{5x-2}\ (5x-2\neq 0)$$

11. The roots of $f(x)-g(x)=0$ are the points where the graphs $f(x)$ and $g(x)$ intersect because $f(x)=g(x)$. On MAPLE we use the commands :

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> f(x):=x^3; g(x):=4+4*x-x^2;
> plot({f(x),g(x)},x=-3..3);
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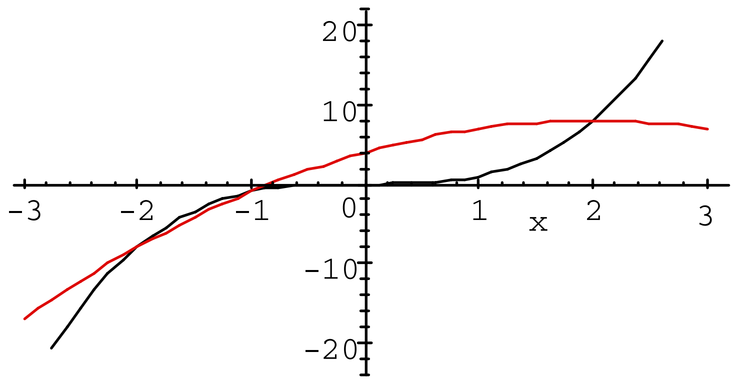
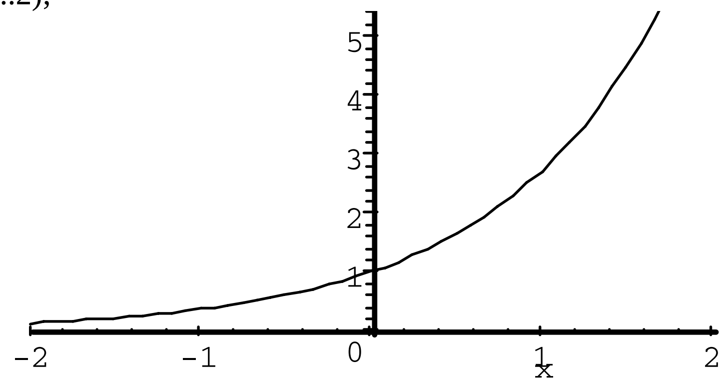


As in solution to question 8 you can plot the graphs closer to the points of intersection. The roots are -2, -1 and 2.

12. (i)> f:=x->2.718281828^x;
> plot(f(x),x=-2..2);



(ii)
> limit((f(h)-1)/h,h=0); shows 0.9999999998 which is equal to 1 correct to two decimal places.
(iii)
> expand((f(x+h)-f(x))/h);
> limit(%,h=0); shows 2.718281828^x. This can also be justified by:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{2.718281828^{x+h}-2.718281828^x}{h}$$
$$=2.718281828^x\underbrace{\lim_{h\to 0}\frac{2.718281828^h-1}{h}}_{=1\text{ by part (ii)}}$$
$$=2.718281828^x$$

13. *Where does the graph cross the t and x axes?*
Crosses the $x(t)$ axis at $t=0$. So we have

$$x(0)=0^2-0-2=-2$$

Crosses the t axis at $x(t)=0$. Thus

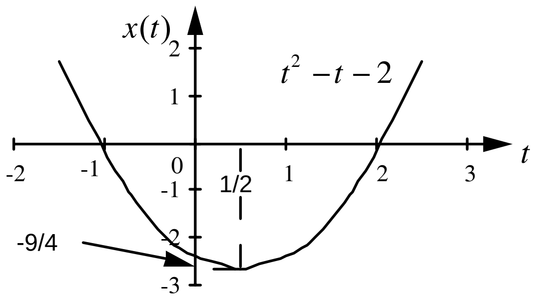
$$t^2-t-2=0$$
$$(t-2)(t+1)=0$$
$$t=2\,,t=-1$$

The shape of the graph t^2-t-2 is similar to the graph of t^2. What adjustments do we need to make to the graph of t^2 to sketch t^2-t-2?
To find the adjustment we need to complete the square on t^2-t-2.

$$t^2 - t - 2 = \left(t - \frac{1}{2}\right)^2 - \frac{1}{4} - 2$$
$$= \left(t - \frac{1}{2}\right)^2 - \frac{9}{4}$$

What are the modifications required on t^2 to obtain the graph of $\left(t - \frac{1}{2}\right)^2 - \frac{9}{4}$?

The $\left(t - \frac{1}{2}\right)^2$ shifts the graph of t^2 by a $\frac{1}{2}$ to the right. The $-\frac{9}{4}$ shifts it down by $\frac{9}{4}$, thus we have:



14. We multiply just like ordinary arithmetical fractions, that is multiply the numerators and multiply the denominators.

$$G(s) = \frac{3(s+1)}{(s+2)(s^2+5s+6)}$$
$$= \frac{3s+3}{(s+2)(s^2+5s+6)} \qquad (*)$$

How do we multiply out the brackets in the denominator? Multiply the first term, s, by s^2+5s+6 and then the second term, 2, by s^2+5s+6.

$$(s+2)(s^2+5s+6) = s(s^2+5s+6) + 2(s^2+5s+6)$$
$$= s^3 + 5s^2 + 6s + 2s^2 + 10s + 12$$
$$= s^3 + \underbrace{5s^2 + 2s^2}_{=7s^2} + \underbrace{6s + 10s}_{=16s} + 12$$
$$= s^3 + 7s^2 + 16s + 12$$

Putting this, $s^3 + 7s^2 + 16s + 12$, for the denominator of (*) gives

$$G(s) = \frac{3s+3}{s^3 + 7s^2 + 16s + 12}$$

15. (a) How do we find the system poles for $G(s) = \dfrac{s^2+1}{s^2+4s+3}$?

They are the values of s where

$$s^2 + 4s + 3 = 0$$
$$(s+3)(s+1) = 0$$
$$s = -3, \; s = -1$$

The system poles are at -3 and -1.

(b) For $G(s) = \dfrac{s+1}{s^2+5s-7}$, the system poles satisfy $s^2 + 5s - 7 = 0$.

How do we find the values of s satisfying this equation? Cannot factorize this equation into simple whole numbers, so we need to use formula (1.16). What are the values of a, b and c for this formula?

For $s^2 + 5s - 7 = 0$; $a = 1$, $b = 5$ and $c = -7$. We have

$$s = \frac{-5 \pm \sqrt{5^2 - (4 \times 1 \times (-7))}}{(2 \times 1)}$$

$$= \frac{-5 \pm \sqrt{53}}{2}$$

$s = 1.14$ and $s = -6.14$ (correct to 2 d.p.)

The poles occur at 1.14 and -6.14.

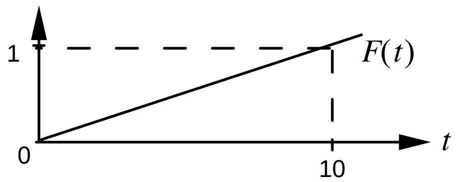
(c) Where are the system poles for $G(s) = \frac{s}{3}$?

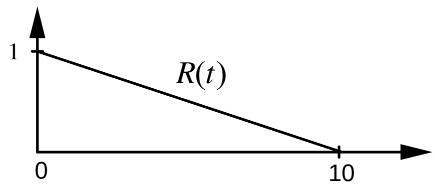
There are **no** system poles because the denominator=3 and therefore cannot be zero.

16. We have $F(t) = \frac{t}{10}$, $R(t) = 1 - \frac{t}{10}$ and

$h(t) = \frac{1/10}{1 - t/10} = \frac{1}{10 - t}$ (multiplying numerator and denominator by 10)

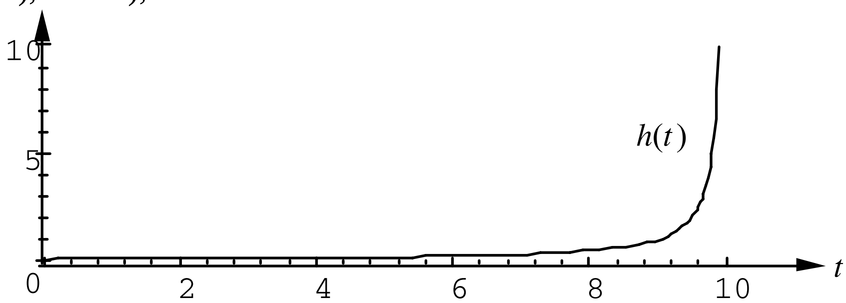
$F(t)$ and $R(t)$ are straight lines of the form $mt + c$:





Use MAPLE to plot $h(t)$ graph for $0 < t < 10$.

plot(1/(10-t),t=0..10);



17. $Q(t)$ is a quadratic, so to sketch $Q(t)$ we could complete the square. Rewriting $Q(t)$ gives

$$-0.015t^2 + 0.3t = -0.015t^2 + \underbrace{(0.015 \times 20)}_{=0.3}t$$

$$= -0.015(t^2 - 20t)$$

$$= -0.015(t - 10)^2 + \underbrace{1.5}_{=(0.015 \times 100)}$$

(1.16) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We need to solve the equation $Q(t) = 0$ which is $0.3t - 0.015t^2 = 0$:

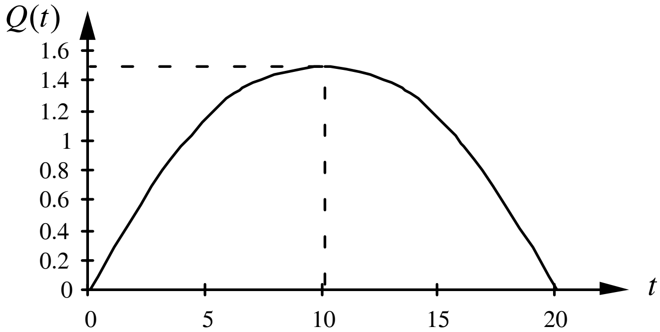
$$t(0.3 - 0.015t) = 0$$

$$t = 0,\ 0.3 - 0.015t = 0$$

$$0.3 = 0.015t$$

$$t = \frac{0.3}{0.015} = 20$$

Hence $Q(t) = 0$ when $t = 0,\ t = 20$.
So $Q(t)$ is a quadratic with a peak at 10 and zeros at 0 and 20:



How do we find $R(t)$?
Substitute $Q(t) = 0.3t - 0.015t^2$ into $R(t) = 1 - Q(t)$:

$$R(t) = 1 - \left(0.3t - 0.015t^2\right)$$

$$= 1 - 0.3t + 0.015t^2$$

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18. Using MAPLE we have:

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> G:=s->(2/(s+1))*(s/((s+2)*(s+5)));
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> simplify(G(s)/(1+0.001*G(s))); shows $1000\dfrac{s}{500s^3 + 4000s + 8501s + 5000}$.