Complete solutions to Miscellaneous Exercise 12

1. The unit vector, $\mathbf{u}$, can be obtained by using (12.5).

$$\mathbf{u}=\frac{1}{\sqrt{5^2+7^2+12^2}}(5\mathbf{i}+7\mathbf{j}+12\mathbf{k})$$
$$=\frac{1}{\sqrt{218}}(5\mathbf{i}+7\mathbf{j}+12\mathbf{k})=\frac{5}{\sqrt{218}}\mathbf{i}+\frac{7}{\sqrt{218}}\mathbf{j}+\frac{12}{\sqrt{218}}\mathbf{k}$$

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2. (i) We have

$$\mathbf{a}+\mathbf{c}=(2\mathbf{i}+3\mathbf{j}-\mathbf{k})+(-\mathbf{i}-\mathbf{j}-6\mathbf{k})$$
$$=(2-1)\mathbf{i}+(3-1)\mathbf{j}+(-1-6)\mathbf{k}=\mathbf{i}+2\mathbf{j}-7\mathbf{k}$$

(ii) Similarly

$$\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=(2\mathbf{i}+3\mathbf{j}-\mathbf{k})+(7\mathbf{i}+12\mathbf{k})+(-\mathbf{i}-\mathbf{j}-6\mathbf{k})+(12\mathbf{i}-\mathbf{j}-11\mathbf{k})$$
$$=(2+7-1+12)\mathbf{i}+(3+0-1-1)\mathbf{j}+(-1+12-6-11)\mathbf{k}=20\mathbf{i}+\mathbf{j}-6\mathbf{k}$$

(iii) $\mathbf{a}\cdot\mathbf{b}=(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(7\mathbf{i}+12\mathbf{k})=(2\times7)+(3\times0)+(-1\times12)=2$

(iv) Since $\mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b}=2$

(v) We first obtain $\mathbf{b}+\mathbf{c}$ and then find the scalar product of $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$:

$$\mathbf{b}+\mathbf{c}=(7\mathbf{i}+12\mathbf{k})+(-\mathbf{i}-\mathbf{j}-6\mathbf{k})=(7-1)\mathbf{i}+(0-1)\mathbf{j}+(12-6)\mathbf{k}=6\mathbf{i}-\mathbf{j}+6\mathbf{k}$$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(6\mathbf{i}-\mathbf{j}+6\mathbf{k})=(2\times6)+(3\times(-1))+(-1\times6)=3$$

(vi) The unit vector, $\hat{\mathbf{a}}$, in the direction of $\mathbf{a}$ can be found by using (12.5).

$$\hat{\mathbf{a}}=\frac{1}{\sqrt{2^2+3^2+(-1)^2}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k})=\frac{1}{\sqrt{14}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$$

Next we find the scalar product $\mathbf{c}\cdot\mathbf{d}$

$$\mathbf{c}\cdot\mathbf{d}=(-\mathbf{i}-\mathbf{j}-6\mathbf{k})\cdot(12\mathbf{i}-\mathbf{j}-11\mathbf{k})=-12+1+66=55$$

Hence we have

$$(\mathbf{c}\cdot\mathbf{d})\hat{\mathbf{a}}=55\frac{1}{\sqrt{14}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k})=\left(\frac{55\times2}{\sqrt{14}}\right)\mathbf{i}+\left(\frac{55\times3}{\sqrt{14}}\right)\mathbf{j}-\left(\frac{55\times1}{\sqrt{14}}\right)\mathbf{k}=29.4\mathbf{i}+44.1\mathbf{j}-14.7\mathbf{k}$$

(vii) We have already found $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$ in part (ii), so the magnitude is

$$|\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}|=|20\mathbf{i}+\mathbf{j}-6\mathbf{k}|=\sqrt{20^2+1^2+(-6)^2}=20.9$$

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3. Let θ be the angle between $\mathbf{v}$ and $\mathbf{a}$. We use

$$\cos(\theta)=\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}||\mathbf{a}|} \qquad (*)$$

Evaluating each of the components of (*):

$$\mathbf{v}\cdot\mathbf{a}=(-4\mathbf{i}-8\mathbf{j}+4\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+4\mathbf{k})=-8-8+16=0$$
$$|\mathbf{v}|=|-4\mathbf{i}-8\mathbf{j}+4\mathbf{k}|=\sqrt{(-4)^2+(-8)^2+4^2}=\sqrt{96}$$
$$|\mathbf{a}|=|2\mathbf{i}+\mathbf{j}+4\mathbf{k}|=\sqrt{2^2+1^2+4^2}=\sqrt{21}$$

Substituting these into (*) gives $\cos(\theta)=0$, taking inverse cos

$$\theta=\cos^{-1}(0)=90°$$

The velocity vector is perpendicular to the acceleration vector.

(12.5) $$\mathbf{u}=\frac{1}{\sqrt{a^2+b^2+c^2}}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$$

4. Similar to solution 3. Use (*) of solution 3:

$$\mathbf{v}\cdot\mathbf{a}=(6\mathbf{i}+2\mathbf{j})\cdot(3\mathbf{i}+7\mathbf{j})=(6\times 3)+(2\times 7)=32$$

$$|\mathbf{v}|=|6\mathbf{i}+2\mathbf{j}|=\sqrt{6^2+2^2}=\sqrt{40}$$

$$|\mathbf{a}|=|3\mathbf{i}+7\mathbf{j}|=\sqrt{3^2+7^2}=\sqrt{58}$$

Substituting these into (*)

$$\cos(\theta)=\frac{32}{\sqrt{40}\sqrt{58}} \quad \text{which gives} \quad \theta=\cos^{-1}\left(\frac{32}{\sqrt{40}\sqrt{58}}\right)=48.37^\circ$$

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5. We first place each force into polar form:

$$18\angle(-45^\circ) \text{ and } 7.6\angle 150^\circ$$

Converting these forces into $\mathbf{i}$ and $\mathbf{j}$ components via calculator

$$18\angle(-45^\circ)=12.73\mathbf{i}-12.73\mathbf{j}$$

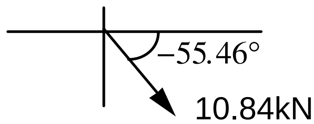
$$7.6\angle 150^\circ=-6.58\mathbf{i}+3.8\mathbf{j}$$

and adding gives the resultant force $\mathbf{R}$ as

$$\mathbf{R}=(12.73-6.58)\mathbf{i}+(-12.73+3.8)\mathbf{j}$$

$$=6.15\mathbf{i}-8.93\mathbf{j}$$

$$=10.84\angle(-55.46^\circ)$$



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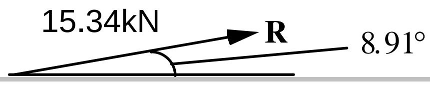
6. Similar to solution 5. Putting each force into polar form

$$15\angle 50^\circ,\ 7.8\angle(-45^\circ) \text{ and } 3.6(-90^\circ)$$

and then using a calculator gives the resultant force $\mathbf{R}$ as

$$\mathbf{R}=15.34\angle 8.91^\circ$$



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7. By using trigonometry we have

$$\mathbf{F}_x=39.5\cos(53^\circ)=23.77\text{ kN} \quad \text{(horizontal)}$$

$$\mathbf{F}_y=39.5\sin(53^\circ)=31.55\text{ kN} \quad \text{(vertically down)}$$

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8. Adding each component $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ separately gives:

(a) $\mathbf{R}=(2-3+61+3)\mathbf{i}+(-1-2+64+8)\mathbf{j}+(7+11+19+11)\mathbf{k}=63\mathbf{i}+69\mathbf{j}+48\mathbf{k}$

(b) $\mathbf{R}=(1+16+47+22)\mathbf{i}+(1+99+84+41)\mathbf{j}+(1+19+88+3)\mathbf{k}=86\mathbf{i}+225\mathbf{j}+111\mathbf{k}$

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9. Using (12.15) we have

$$\text{work done }=(\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(15\mathbf{i}+20\mathbf{j})=15+20=35J$$

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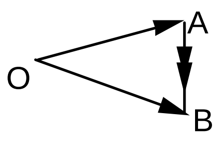
10. We have $\vec{OA}=2\mathbf{i}+3\mathbf{j}+7\mathbf{k}$ and $\vec{OB}=7\mathbf{i}-11\mathbf{j}+37\mathbf{k}$



$$\vec{AB}=\vec{OB}-\vec{OA}$$

Thus

$$\vec{AB}=\vec{OB}-\vec{OA}=(7\mathbf{i}-11\mathbf{j}+37\mathbf{k})-(2\mathbf{i}+3\mathbf{j}+7\mathbf{k})=5\mathbf{i}-14\mathbf{j}+30\mathbf{k}$$

(12.15) $\quad \text{work done }=\mathbf{F}\cdot\mathbf{r}$

Applying (12.15) gives

$$\text{work done} = (2\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}) \cdot (5\mathbf{i} - 14\mathbf{j} + 30\mathbf{k})$$

$$= (2 \times 5) + (-7 \times (-14)) + (12 \times 30) = 468 \text{ J}$$

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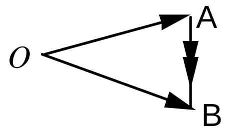
11. (a) The position vectors in $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components of A and B are

$$\vec{OA} = \mathbf{i} + 7\mathbf{j} + \mathbf{k}, \ \vec{OB} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$



$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= (3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) - (\mathbf{i} + 7\mathbf{j} + \mathbf{k})$$

$$= 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

By (12.15) we have $\text{work done} = (3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 6 + 3 + 4 = 13 \text{ J}$.

(b) Similarly we have

$$\vec{OA} = 3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}, \ \vec{OB} = 6\mathbf{i} + 7\mathbf{j} - 17\mathbf{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = (6\mathbf{i} + 7\mathbf{j} - 17\mathbf{k}) - (3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) = 3\mathbf{i} + 3\mathbf{j} - 20\mathbf{k}$$

By (12.15)

$$\text{work done} = (2\mathbf{i} + \mathbf{j} - 7\mathbf{k}) \cdot (3\mathbf{i} + 3\mathbf{j} - 20\mathbf{k}) = 6 + 3 + (-7 \times (-20)) = 149 \text{ J}$$

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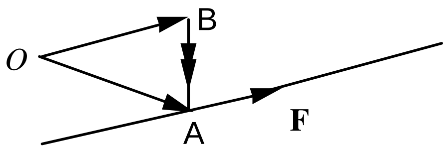
12. We have



$$\vec{BA} = \vec{OA} - \vec{OB}$$

$$= (3\mathbf{i} + \mathbf{j} + 8\mathbf{k}) - (2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k})$$

$$= \mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

Let $\mathbf{M}$ be the moment then by (12.25)

$$\mathbf{M} = (\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \times (5\mathbf{i} - \mathbf{j} - \mathbf{k}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 1 \\ 5 & -1 & -1 \end{pmatrix}$$

$$= \mathbf{i}\left[\det \begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix}\right] - \mathbf{j}\left[\det \begin{pmatrix} 1 & 1 \\ 5 & -1 \end{pmatrix}\right] + \mathbf{k}\left[\det \begin{pmatrix} 1 & 4 \\ 5 & -1 \end{pmatrix}\right]$$

$$= \mathbf{i}(-4 + 1) - \mathbf{j}(-1 - 5) + \mathbf{k}(-1 - 20) = -3\mathbf{i} + 6\mathbf{j} - 21\mathbf{k}$$

The magnitude of $\mathbf{M}$ is

$$|\mathbf{M}| = |-3\mathbf{i} + 6\mathbf{j} - 21\mathbf{k}| = \sqrt{(-3)^2 + 6^2 + (-21)^2} = 22.05 \text{ N m}$$

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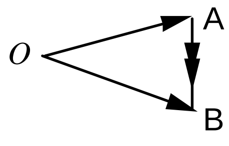
13. We have $\vec{OA} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\vec{OB} = 5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$



$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= (5\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$= (5 + 1)\mathbf{i} + (1 - 2)\mathbf{j} + (-2 - 3)\mathbf{k} = 6\mathbf{i} - \mathbf{j} - 5\mathbf{k}$$

The unit vector, $\mathbf{u}$, in the direction of $\vec{AB}$ is obtained by using:

(12.15) Work done $= \mathbf{F} \cdot \mathbf{r}$

(12.25) Moment $= \mathbf{r} \times \mathbf{F}$

$$\mathbf{u}=\frac{1}{\sqrt{6^2+(-1)^2+(-5)^2}}(6\mathbf{i}-\mathbf{j}-5\mathbf{k})=\frac{1}{\sqrt{62}}(6\mathbf{i}-\mathbf{j}-5\mathbf{k})$$

Thus

$$\mathbf{F}=\frac{39}{\sqrt{62}}(6\mathbf{i}-\mathbf{j}-5\mathbf{k})=(29.72\mathbf{i}-4.95\mathbf{j}-24.77\mathbf{k})\text{ kN}$$

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14. We need to find

$$(\mathbf{F}_1\times\mathbf{r}_1)+(\mathbf{F}_2\times\mathbf{r}_2)+(\mathbf{F}_3\times\mathbf{r}_3) \qquad (\dagger)$$

Each vector product is evaluated by using (12.24):

$$\mathbf{F}_1\times\mathbf{r}_1=(\mathbf{i}+3\mathbf{j}-6\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$

$$=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&3&-6\\2&2&2\end{pmatrix}=\mathbf{i}\left[\det\begin{pmatrix}3&-6\\2&2\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}1&-6\\2&2\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}1&3\\2&2\end{pmatrix}\right]$$

$$=\mathbf{i}(18)-\mathbf{j}(14)+\mathbf{k}(-4)=18\mathbf{i}-14\mathbf{j}-4\mathbf{k}$$

Similarly

$$\mathbf{F}_2\times\mathbf{r}_2=(-7\mathbf{i}+2\mathbf{j}+5\mathbf{k})\times(3\mathbf{i}-2\mathbf{j}-\mathbf{k})$$

$$=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-7&2&5\\3&-2&-1\end{pmatrix}=\mathbf{i}\left[\det\begin{pmatrix}2&5\\-2&-1\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}-7&5\\3&-1\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}-7&2\\3&-2\end{pmatrix}\right]$$

$$=\mathbf{i}(8)-\mathbf{j}(-8)+\mathbf{k}(8)=8\mathbf{i}+8\mathbf{j}+8\mathbf{k}$$

Also

$$\mathbf{F}_3\times\mathbf{r}_3=(2\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}-10\mathbf{k})$$

$$=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&2&3\\2&2&-10\end{pmatrix}=\mathbf{i}\left[\det\begin{pmatrix}2&3\\2&-10\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}0&3\\2&-10\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}0&2\\2&2\end{pmatrix}\right]$$

$$=\mathbf{i}(-26)-\mathbf{j}(-6)+\mathbf{k}(-4)=-26\mathbf{i}+6\mathbf{j}-4\mathbf{k}$$

Substituting these into $(\dagger)$:

$$(18\mathbf{i}-14\mathbf{j}-4\mathbf{k})+(8\mathbf{i}+8\mathbf{j}+8\mathbf{k})+(-26\mathbf{i}+6\mathbf{j}-4\mathbf{k})=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$$

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15. We have by (12.24)

$$\mathbf{r}\times\mathbf{s}=(a\mathbf{i}+b\mathbf{k})\times(c\mathbf{i}+d\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a&0&b\\c&0&d\end{pmatrix}$$

$$=\mathbf{i}\left[\det\begin{pmatrix}0&b\\0&d\end{pmatrix}\right]-\mathbf{j}\left[\det\begin{pmatrix}a&b\\c&d\end{pmatrix}\right]+\mathbf{k}\left[\det\begin{pmatrix}a&0\\c&0\end{pmatrix}\right]$$

$$=0-\mathbf{j}(ad-cb)+0=\mathbf{j}(cb-ad)$$

(12.24)
$$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times(d\,\mathbf{i}+e\mathbf{j}+f\,\mathbf{k})=\det\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a&b&c\\d&e&f\end{pmatrix}$$

16. Since the vectors are $90°$ to each other we have

$$\cos(90°) = 0 = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Thus the numerator, $\mathbf{a}\cdot\mathbf{b}$, must be zero.

$$\mathbf{a}\cdot\mathbf{b} = (2\mathbf{i}-3\mathbf{j}+x\mathbf{k})\cdot(-3\mathbf{i}+\mathbf{j}+x\mathbf{k}) = -6-3+x^2 = 0$$

Rearranging and solving the quadratic

$$x^2-9=0$$

$$x^2=9$$

$$x=\sqrt{9}=-3,\ 3$$

The vectors $\mathbf{a}$ and $\mathbf{b}$ are $90°$ to each other if $x=-3$ or $x=3$.

17. We need to write the exact value of $\cos(45°)$:

$$\cos(45°) = \frac{1}{\sqrt{2}}$$

We have

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{1}{\sqrt{2}}$$

Rearranging gives

$$\sqrt{2}(\mathbf{a}\cdot\mathbf{b}) = |\mathbf{a}||\mathbf{b}| \qquad (*)$$

We evaluate each component of (*)

$$\mathbf{a}\cdot\mathbf{b} = (2\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(x\mathbf{i}+\mathbf{j}-\mathbf{k}) = 2x+2-1 = 2x+1$$

$$|\mathbf{a}| = |2\mathbf{i}+2\mathbf{j}+\mathbf{k}| = \sqrt{2^2+2^2+1^2} = 3$$

$$|\mathbf{b}| = |x\mathbf{i}+\mathbf{j}-\mathbf{k}| = \sqrt{x^2+1^2+(-1)^2} = \sqrt{x^2+2}$$

Substituting these into (*) gives

$$\sqrt{2}(2x+1) = 3\sqrt{x^2+2}$$

Squaring both sides

$$2(2x+1)^2 = 9(x^2+2)$$

$$2(4x^2+4x+1) = 9x^2+18$$

$$8x^2+8x+2 = 9x^2+18$$

Rearranging

$$9x^2-8x^2-8x+18-2=0$$

$$x^2-8x+16=0$$

$$(x-4)^2=0 \text{ gives } x=4$$

Hence $x=4$ gives an angle of $45°$ between the two vectors.