Complete solutions to Miscellaneous Exercise 14

1. (a) Characteristic equation is

$$m^2+8m+16=0$$
$$(m+4)^2=0$$
$$m=-4$$

By (14.5) $y=(A+Bx)e^{-4x}$

(b) Characteristic equation is

$$m^2-2m-3=0$$
$$(m-3)(m+1)=0$$
$$m_1=3,\ m_2=-1$$

By (14.4) $y=Ae^{3x}+Be^{-x}$

(c) Characteristic equation is

$$m^2-6m+7=0$$

Solving this quadratic equation gives the roots $3+\sqrt{2}$ and $3-\sqrt{2}$. Hence by (14.4)

$$y=Ae^{(3+\sqrt{2})x}+Be^{(3-\sqrt{2})x}$$

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2. Note the complementary function is the same in each case, (a) - (c), and it is the solution to question 1(a),

$$y_c=(A+Bx)e^{-4x}$$

(a) Since $f(x)=8$, so our trial function is a constant

$$Y=C$$

Substituting into $\frac{d^2Y}{dx^2}+\frac{dY}{dx}+16Y=8$ yields

$$16C=8 \text{ gives } C=\frac{1}{2}$$

The general solution, $y=y_c+Y$, is $y=(A+Bx)e^{-4x}+\frac{1}{2}$

(b) By (14.12) $Y=ax+b$

$$\frac{dY}{dx}=a,\ \frac{d^2Y}{dx^2}=0$$

Substituting into $\frac{d^2Y}{dx^2}+8\frac{dY}{dx}+16Y=8x$ gives

$$0+8a+16(ax+b)=8x$$

Equating coefficients of x:

$$16a=8 \text{ gives } a=\frac{1}{2}$$

constants: $8a+16b=0,\ 4+16b=0$

$$16b=-4 \text{ gives } b=-\frac{1}{4}$$

(14.4)	m_1 and m_2 gives $y=Ae^{m_1x}+Be^{m_2x}$
(14.5)	m (equal roots) gives $y=(A+Bx)e^{mx}$
(14.12)	If $f(x)=Ax+B$ then $Y=ax+b$

$$Y=\frac{1}{2}x-\frac{1}{4}=\frac{1}{4}(2x-1)$$

Hence $\quad y=(A+Bx)e^{-4x}+\frac{1}{4}(2x-1)$

(c) Since $f(x)=x^2+x+1$, a quadratic, so our trial function is $Y=ax^2+bx+c$

$$\frac{dY}{dx}=2ax+b,\ \frac{d^2Y}{dx^2}=2a$$

Substituting into $\frac{d^2Y}{dx^2}+8\frac{dY}{dx}+16Y=x^2+x+1$ gives

$$2a+8(2ax+b)+16(ax^2+bx+c)=x^2+x+1$$

Equating coefficients of

x^2: $\quad 16a=1$ gives $a=\frac{1}{16}$

x: $\quad 16a+16b=1$ gives $b=0$

constants: $\quad 2a+8b+16c=1$

$$\frac{2}{16}+16c=1 \text{ gives } c=\frac{14}{16^2}$$

Substituting $a=\frac{1}{16}$, $b=0$ and $c=\frac{14}{16^2}$ into $Y=ax^2+bx+c$ gives

$$Y=\frac{1}{16}x^2+\frac{14}{16^2}=\frac{1}{16^2}\left[16x^2+14\right]$$

$$y=(A+Bx)e^{-4x}+\frac{1}{256}\left[16x^2+14\right]$$

Hence $\quad =(A+Bx)e^{-4x}+\frac{1}{128}\left[8x^2+7\right]$

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3. The characteristic equation is given by

$$m^2+\frac{GJ}{Il}=0$$

$$m^2+\left(\sqrt{\frac{GJ}{Il}}\right)^2=0$$

By (14.8) we have $\theta=A\cos(\omega t)+B\sin(\omega t)$ where $\omega=\sqrt{\frac{GJ}{Il}}$

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4. Characteristic equation is

$$m^2-\frac{hP}{kA}=0$$

$$m^2-\left(\sqrt{\frac{hP}{kA}}\right)^2=0$$

By (14.9) we have $T=Ae^{mx}+Be^{-mx}$ where $m=\sqrt{\frac{hP}{kA}}$

(14.8) $\quad m^2+k^2=0$ gives $y=A\cos(kx)+B\sin(kx)$

(14.9) $\quad m^2-k^2=0$ gives $y=Ae^{kx}+Be^{-kx}$

5. Rearranging the given differential equation.

$$m\ddot{x}+kx=F\cos(\omega t) \qquad (*)$$

By solution to question 5 of **EXERCISE 14(c)**, the complementary function is

$$x_c=A\cos(\omega t)+B\sin(\omega t) \qquad \omega=\sqrt{k/m}$$

Particular integral X;
Using TABLE 1 gives the trial function to be

$$a\cos(\omega t)+b\sin(\omega t)$$

but this is already part of the complementary function, so use the trial function

$$X=\left[a\cos(\omega t)+b\sin(\omega t)\right]t \qquad (\dagger)$$

By differentiating this twice and substituting into $\ddot{X}+\frac{k}{m}X=\frac{F}{m}\cos(\omega t)$ we obtain

$$2\omega\left[b\cos(\omega t)-a\sin(\omega t)\right]=\frac{F}{m}\cos(\omega t)$$

Equating coefficients of

$\sin(\omega t)$: $\quad -2\omega a=0,$ hence $a=0$

$\cos(\omega t)$: $\quad 2\omega b=\frac{F}{m},$ hence $b=\frac{F}{2\omega m}$

Substituting $a=0$ and $b=\frac{F}{2\omega m}$ into $X=\left[a\cos(\omega t)+b\sin(\omega t)\right]t$ gives

$$X=\frac{Ft}{2\omega m}\sin(\omega t)$$

Since $x=x_c+X$, so the general solution is

$$x=A\cos(\omega t)+B\sin(\omega t)+\frac{Ft}{2\omega m}\sin(\omega t)$$

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6. Dividing the given differential equation by L gives $\ddot{\theta}+\frac{g}{L}\theta=0$

Characteristic equation is

$$m^2+\frac{g}{L}=0$$

By (14.8)

$$\theta=A\cos(\omega t)+B\sin(\omega t) \qquad (*)$$

where $\omega=\sqrt{\frac{g}{L}}$. Substituting the initial condition, when $t=0,\ \theta=1$;

$$1=A\cos(0)+B\sin(0) \quad \text{gives} \quad 1=A$$

Differentiating (*) gives

$$\dot{\theta}=-\omega A\sin(\omega t)+\omega B\cos(\omega t)$$

(14.8) $\quad m^2+k^2=0$ gives $\theta=A\cos(kt)+B\sin(kt)$

Substituting the other initial condition, when $t=0$, $\dot{\theta}=\sqrt{3}\omega$;

$$\sqrt{3}\omega=-\omega A\sin(0)+\omega B\cos(0) \text{ gives } \sqrt{3}\omega=\omega B,\ B=\sqrt{3}$$

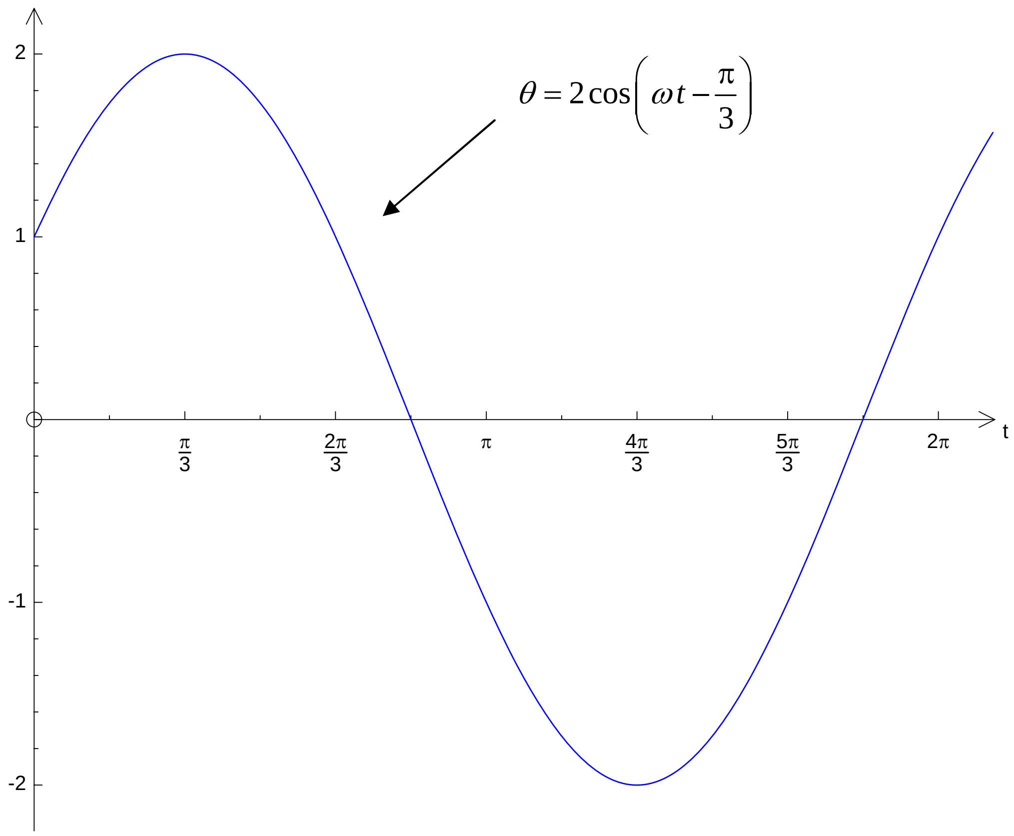
Placing $A=1$ and $B=\sqrt{3}$ into (*) gives

$$\theta=\cos(\omega t)+\sqrt{3}\sin(\omega t)$$

By using (4.75) we have

$$r=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2 \text{ and } \alpha=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$$

Hence using (4.75) on $\theta=\cos(\omega t)+\sqrt{3}\sin(\omega t)$ gives $\theta=2\cos\left(\omega t-\frac{\pi}{3}\right)$



(4.75) $\quad A\cos(\omega t)+B\sin(\omega t)=\sqrt{A^2+B^2}\cos(\omega t-\alpha)$ where $\alpha=\tan^{-1}\left(\frac{B}{A}\right)$

7. The characteristic equation is $Cm^2 + \frac{1}{L} = 0$

Dividing through by C

$$m^2 + \frac{1}{LC} = 0$$

$$m^2 + \omega^2 = 0 \text{ where } \omega^2 = \frac{1}{LC}$$

We have

$$v = A\cos(\omega t) + B\sin(\omega t)$$

$$v \underbrace{=}_{\text{by (4.75)}} \sqrt{A^2 + B^2}\cos(\omega t - \alpha) \quad \text{where } \alpha = \tan^{-1}\left(\frac{B}{A}\right)$$

$$v = r\cos(\omega t - \alpha) \qquad \left(r = \sqrt{A^2 + B^2}\right)$$

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8. (i) Characteristic equation is given by $Cm^2 + \frac{1}{R}m + \frac{1}{L} = 0$

Dividing by C

$$m^2 + \frac{1}{RC}m + \frac{1}{LC} = 0$$

(ii) Equating with $m^2 + 2\zeta\omega m + \omega^2 = 0$ so that we have

$$m^2 + 2\zeta\omega m + \omega^2 = m^2 + \frac{1}{RC}m + \frac{1}{LC}$$

$$\omega^2 = \frac{1}{LC} \text{ gives } \omega = \frac{1}{\sqrt{LC}}$$

Equating the m terms gives

$$2\zeta\omega = \frac{1}{RC}$$

For critical resistance $\zeta = 1$, so we have $R = R_{cr}$ and

$$2\omega = \frac{1}{R_{cr}C}$$

Substituting

$$\omega = \frac{1}{\sqrt{LC}}$$

into $2\omega = \frac{1}{R_{cr}C}$ gives

$$2\frac{1}{\sqrt{LC}} = \frac{1}{R_{cr}C}$$

$$R_{cr} = \frac{1}{2}\frac{\sqrt{LC}}{C} = \frac{1}{2}\frac{\sqrt{L}\sqrt{C}}{C} = \frac{1}{2}\sqrt{\frac{L}{C}} \text{ (using the rules of indices)}$$

$$R_{cr} = \frac{1}{2}\sqrt{\frac{L}{C}}$$

9. (i) Characteristic equation is

$$Lm^2 + Rm + \frac{1}{C} = 0$$

Dividing by L;

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

(ii) Equating

$$m^2 + 2\zeta\omega m + \omega^2 = m^2 + \frac{R}{L}m + \frac{1}{LC}$$

Equating constants (without m)

$$\omega^2 = \frac{1}{LC} \text{ gives } \omega = \frac{1}{\sqrt{LC}}$$

Equating the m terms

$$2\zeta\omega = \frac{R}{L}$$

For critical resistance, $\zeta = 1$, hence

$$\frac{R_{cr}}{L} = 2\omega$$

$$R_{cr} = 2\omega L = 2\frac{1}{\sqrt{LC}}L = 2\sqrt{\frac{L}{C}}$$

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10. (i) We are given $L = 1\times 10^{-3}$ and $C = 10\times 10^{-6}$. By solution 9 we have

$$R_{cr} = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{1\times 10^{-3}}{10\times 10^{-6}}} = 20$$

(R_{cr} is the critical resistance).

(ii) Substituting $R = 20$, $L = 1\times 10^{-3}$ and $C = 10\times 10^{-6}$ into the homogeneous equation gives

$$\left(1\times 10^{-3}\right)\frac{d^2i}{dt^2} + 20\frac{di}{dt} + \left(\frac{1}{10\times 10^{-6}}\right)i = 0$$

Dividing by 1×10^{-3} yields

$$\frac{d^2i}{dt^2} + \left(\frac{20}{1\times 10^{-3}}\right)\frac{di}{dt} + \left(\frac{1}{10\times 10^{-9}}\right)i = 0$$

$$\frac{d^2i}{dt^2} + \left(20\times 10^{3}\right)\frac{di}{dt} + \left(1\times 10^{8}\right)i = 0$$

The characteristic equation is

$$m^2 + \left(20\times 10^3\right)m + \left(1\times 10^8\right) = 0$$

To find m we use the quadratic equation formula, (1.16), with $a = 1$, $b = 20\times 10^3$ and $c = 1\times 10^8$

$$m = \frac{-\left(20\times 10^3\right) \pm \sqrt{\left(20\times 10^3\right)^2 - \left(4\times 10^8\right)}}{2} = -10\times 10^3$$

(1.16) $$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since we have equal roots so by (14.5) we have

$$i = (A + Bt)e^{-(10\times10^3)t}$$

11. Dividing the given differential equation by C

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

Characteristic equation is

$$m^2 + \frac{1}{RC}m + \frac{1}{LC} = 0$$

Substituting the given values of $R = 500,\ L = 100\times10^{-3}$ and $C = 0.5\times10^{-6}$ gives

$$m^2 + \left(\frac{1}{500\times0.5\times10^{-6}}\right)m + \left(\frac{1}{100\times10^{-3}\times0.5\times10^{-6}}\right) = 0$$

$$m^2 + 4000m + (2\times10^7) = 0$$

Putting $a = 1,\ b = 4000$ and $c = 2\times10^7$ into (1.16) gives

$$m = \frac{-4000 \pm \sqrt{4000^2 - (4\times2\times10^7)}}{2}$$

$$= -2000 \pm \frac{1}{2}\sqrt{-64000000} = -2000 \pm j4000$$

By (14.6)

$$v = e^{-2000t}\left[A\cos(4000t) + B\sin(4000t)\right] \qquad (*)$$

Substituting $t = 0,\ v = 9$ gives

$$9 = A$$

Need to differentiate by using the product rule, (6.31),

$$\frac{dv}{dt} = -2000e^{-2000t}\left[A\cos(4000t) + B\sin(4000t)\right] + e^{-2000t}\left[-4000A\sin(4000t) + 4000B\cos(4000t)\right]$$

Substituting the other initial condition, when $t = 0,\ \frac{dv}{dt} = 0$ and $A = 9$;

$$0 = -2000(9+0) + (0 + 4000B)$$

$$4000B = 2000\times9 \text{ gives } B = 4.5$$

Substituting $A = 9$ and $B = 4.5$ into (*) yields

$$v = e^{-2000t}\left[9\cos(4000t) + 4.5\sin(4000t)\right]$$

12. By solution 4, $T = Ce^{mx} + De^{-mx}$ (†)

where $m = \sqrt{\frac{hP}{kA}} \neq 0$, C and D are constants. Substituting x=0, $T = T_B$;

$$C + D = T_B, \text{ hence } C = T_B - D \qquad (*)$$

Differentiating and substituting $x = L,\ \frac{dT}{dx} = 0$ gives

(14.5) m (equal roots) then $y = (A + Bx)e^{mx}$

(14.6) $m = \alpha \pm j\beta$ then $y = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$

Since m does **not** equal zero therefore $\frac{dT}{dx} = m\left(Ce^{mL} - De^{-mL}\right) = 0$

$$Ce^{mL} - De^{-mL} = 0$$

Substituting $C = T_B - D$ gives

$$\left(T_B - D\right)e^{mL} - De^{-mL} = 0$$

$$T_B e^{mL} = D\left(e^{mL} + e^{-mL}\right)$$

$$D = \frac{T_B e^{mL}}{e^{mL} + e^{-mL}}$$

From (*), $C = T_B - D$, we have

$$C = T_B - \frac{T_B e^{mL}}{e^{mL} + e^{-mL}}$$

$$= \frac{T_B\left(e^{mL} + e^{-mL}\right) - T_B e^{mL}}{e^{mL} + e^{-mL}} = T_B\left(\frac{e^{mL} + e^{-mL} - e^{mL}}{e^{mL} + e^{-mL}}\right)$$

$$C = \frac{T_B e^{-mL}}{e^{mL} + e^{-mL}}$$

Substituting $C = \frac{T_B e^{-mL}}{e^{mL} + e^{-mL}}$ and $D = \frac{T_B e^{mL}}{e^{mL} + e^{-mL}}$ into $T = Ce^{mx} + De^{-mx}$ gives

$$T = \frac{T_B e^{-mL} e^{mx} + T_B e^{mL} e^{-mx}}{e^{mL} + e^{-mL}}$$

$$= T_B\left(\frac{e^{mL-mx} + e^{-mL+mx}}{e^{mL} + e^{-mL}}\right) = T_B\left(\frac{e^{m(L-x)} + e^{-m(L-x)}}{e^{mL} + e^{-mL}}\right)$$

Using (5.24) gives

$$T = T_B \frac{\cosh\left[m\left(L - x\right)\right]}{\cosh\left(mL\right)}$$

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13. Similar to solution 12. We have

$$T = Ce^{mx} + De^{-mx} \qquad (\dagger)$$

where m (as in solution 12), C and D are constants. Substituting $x = 0,\ \ T = T_B$ into $(\dagger)$ gives

$$T_B = C + D \quad \text{hence} \quad D = T_B - C$$

Putting $x = L,\ T = 0$ into $(\dagger)$ gives

$$0 = Ce^{mL} + De^{-mL} = Ce^{mL} + \left(T_B - C\right)e^{-mL}$$

$$0 = C\left(e^{mL} - e^{-mL}\right) + T_B e^{-mL}$$

Hence $C = \frac{-T_B e^{-mL}}{e^{mL} - e^{-mL}}$. Substituting this C into $D = T_B - C$ gives

$$D = T_B + \frac{T_B e^{-mL}}{e^{mL} - e^{-mL}} = T_B\left(\frac{e^{mL} - e^{-mL} + e^{-mL}}{e^{mL} - e^{-mL}}\right)$$

(5.24) $\qquad \cosh\left(x\right) = \frac{e^{x} + e^{-x}}{2}$

$$D = T_B\left(\frac{e^{mL}}{e^{mL} - e^{-mL}}\right)$$

Substituting $C = -\frac{T_B e^{-mL}}{e^{mL} - e^{-mL}}$ and $D = \frac{T_B e^{mL}}{e^{mL} - e^{-mL}}$ into $T = De^{-mx} + Ce^{mx}$ gives

$$T = \left(\frac{T_B e^{mL} e^{-mx}}{e^{mL} - e^{-mL}}\right) - \left(\frac{T_B e^{-mL} e^{mx}}{e^{mL} - e^{-mL}}\right)$$

$$= T_B\left(\frac{e^{m(L-x)} - e^{-m(L-x)}}{e^{mL} - e^{-mL}}\right)$$

$$T \underbrace{=}_{\text{by (5.23)}} T_B \frac{\sinh\left[m(L-x)\right]}{\sinh(mL)}$$

14. By **EXAMPLE 6** we have

$$y = A\cos(kx) + B\sin(kx) \quad \text{where } k = \sqrt{P/EI}$$

Substituting $x = 0,\ y = e$ gives

$$e = A\cos(0) + B\sin(0) \quad \text{hence } A = e$$

Substituting $x = L,\ y = e$ and $A = e$ yields

$$e = e\cos(kL) + B\sin(kL)$$

$$B\sin(kL) = e\left[1 - \cos(kL)\right]$$

$$B = e\left(\frac{1 - \cos(kL)}{\sin(kL)}\right) \underbrace{=}_{\text{by hint}} e\tan\left(\frac{kL}{2}\right)$$

Substituting $A = e$ and $B = e\tan\left(\frac{kL}{2}\right)$ into $y = A\cos(kx) + B\sin(kx)$ gives

$$y = e\cos(kx) + e\tan\left(\frac{kL}{2}\right)\sin(kx) = e\left(\cos(kx) + \tan\left(\frac{kL}{2}\right)\sin(kx)\right)$$

15. The given differential equation can be rearranged to

$$EI\frac{d^2y}{dx^2} + Py = \frac{w}{2}\left(x^2 - Lx\right)$$

Dividing by EI

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{w}{2EI}\left(x^2 - Lx\right) \qquad (*)$$

Complementary function, y_c, satisfies $\frac{d^2y_c}{dx^2} + \frac{P}{EI}y_c = 0$

The characteristic equation is

$$m^2 + \frac{P}{EI} = 0$$

$$m^2 + k^2 = 0 \quad \text{where } k^2 = \frac{P}{EI}$$

(5.23) $\sinh(x) = \frac{e^x - e^{-x}}{2}$

(14.8) If $m^2 + k^2 = 0$ then $y = A\cos(kx) + B\sin(kx)$

By (14.8) $y_c = A\cos(kx) + B\sin(kx)$ where k=$\sqrt{P/EI}$.

Particular integral Y:

Since $f(x) = \frac{w}{2EI}\left(x^2 - Lx\right)$ is a quadratic so we use the trial function to be the general quadratic

$$Y = ax^2 + bx + c$$

Differentiating this Y yields

$$\frac{dY}{dx} = 2ax + b,\ \frac{d^2Y}{dx^2} = 2a$$

Substituting these into $\frac{d^2Y}{dx^2} + \frac{P}{EI}Y = \frac{w}{2EI}\left(x^2 - Lx\right)$ gives

$$2a + \frac{P}{EI}\left(ax^2 + bx + c\right) = \frac{w}{2EI}\left(x^2 - Lx\right)$$

Equating coefficients of

x^2: $\quad \frac{Pa}{EI} = \frac{w}{2EI}$ gives $a = \frac{w}{2P}$

x: $\quad \frac{Pb}{EI} = -\frac{wL}{2EI}$ gives $b = -\frac{wL}{2P}$

constants:

$$2a + \frac{Pc}{EI} = 0$$

$$\frac{Pc}{EI} = -2a = -\frac{2w}{2P} \quad \left(\text{substituting } a = \frac{w}{2P}\right)$$

$$c = -\frac{wEI}{P^2}$$

Substituting $a = \frac{w}{2P}$, $b = -\frac{wL}{2P}$ and $c = -\frac{wEI}{P^2}$ into $Y = ax^2 + bx + c$ gives

$$Y = \frac{w}{2P}x^2 - \frac{wL}{2P}x - \frac{wEI}{P^2}$$

$$= \frac{w}{2P}\left(x^2 - Lx - \frac{2EI}{P}\right)$$

$$= \frac{w}{2P}\left(x^2 - Lx - \frac{2}{k^2}\right) \quad \text{where } k^2 = \frac{P}{EI}$$

The general solution is given by $y = y_c + Y$, thus

$$y = A\cos(kx) + B\sin(kx) + \frac{w}{2P}\left(x^2 - Lx - \frac{2}{k^2}\right)$$

(ii) Using the solution obtained in part (i)

$$y = A\cos(kx) + B\sin(kx) + \frac{w}{2P}\left(x^2 - Lx - \frac{2}{k^2}\right) \qquad (\dagger)$$

Substituting the first boundary condition x=0, y=0 into (†) gives

$$0 = A + 0 + \frac{w}{2P}\left(0 - 0 - \frac{2}{k^2}\right)$$

$$A = \frac{2w}{2Pk^2} = \frac{w}{Pk^2}$$

Substituting the other boundary condition $x = L,\ y = 0$ into (†) gives

$$0 = A\cos(kL) + B\sin(kL) + \frac{w}{2P}\left(L^2 - L^2 - \frac{2}{k^2}\right)$$

$$0 = \frac{w}{Pk^2}\cos(kL) + B\sin(kL) - \frac{2w}{2Pk^2}$$

$$\frac{w}{Pk^2} - \frac{w}{Pk^2}\cos(kL) = B\sin(kL)$$

$$\frac{w}{Pk^2}\left[1 - \cos(kL)\right] = B\sin(kL)$$

$$B = \frac{w}{Pk^2}\left[\frac{1 - \cos(kL)}{\sin(kL)}\right]$$

$$= \frac{w}{Pk^2}\tan\left(\frac{kL}{2}\right)$$

The last step is made by using the given identity $\frac{1 - \cos(x)}{\sin(x)} = \tan\left(\frac{x}{2}\right)$.

Substituting $A = \frac{w}{Pk^2}$ and $B = \frac{w}{Pk^2}\tan\left(\frac{kL}{2}\right)$ into (†) gives

$$y = \frac{w}{Pk^2}\cos(kx) + \frac{w}{Pk^2}\tan\left(\frac{kL}{2}\right)\sin(kx) + \frac{w}{2P}\left[x^2 - Lx - \frac{2}{k^2}\right]$$

$$y = \frac{w}{Pk^2}\left[\cos(kx) + \tan\left(\frac{kL}{2}\right)\sin(kx)\right] + \frac{w}{2P}\left[x^2 - Lx - \frac{2}{k^2}\right]$$

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16. Rearrange the given differential equation $\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{1}{EI}(Fx + M)$

Complementary function $y_c = A\cos(kx) + B\sin(kx)$ where $k = \sqrt{\frac{P}{EI}}$

Particular integral Y; since we have a linear function on the right hand side so

$$Y = ax + b,\ \frac{dY}{dx} = a,\ \frac{d^2Y}{dx^2} = 0$$

Substituting these into $\frac{d^2Y}{dx^2} + \frac{P}{EI}Y = -\frac{1}{EI}(Fx + M)$ gives

$$\frac{P}{EI}(ax + b) = -\frac{1}{EI}(Fx + M)$$

Clearly $ax + b = -\frac{1}{P}(Fx + M) = Y$. Since $y = y_c + Y$ we have

$$y = A\cos(kx) + B\sin(kx) - \frac{1}{P}(Fx + M) \qquad (*)$$

Placing $x = 0,\ y = 0$ into (*) gives

$$0 = A + 0 - \frac{1}{P}(0 + M), \text{ hence } \quad A = \frac{M}{P}$$

Substituting $x = L,\ y = 0$ and $A = \frac{M}{P}$ into (*) gives

$$0 = \frac{M}{P}\cos(kL) + B\sin(kL) - \frac{1}{P}(FL + M)$$

$$\frac{M}{P} - \frac{M}{P}\cos(kL) + \frac{FL}{P} = B\sin(kL)$$

$$\frac{M}{P}\left[1 - \cos(kL)\right] + \frac{FL}{P} = B\sin(kL)$$

$$B = \frac{M}{P}\left[\frac{1-\cos(kL)}{\sin(kL)}\right] + \frac{FL}{P\sin(kL)} = \frac{M}{P}\underbrace{\tan\left(\frac{kL}{2}\right)}_{\text{by hint}} + \frac{FL}{P}\operatorname{cosec}(kL)$$

Substituting $A = \frac{M}{P}$ and $B = \frac{M}{P}\tan\left(\frac{kL}{2}\right) + \frac{FL}{P}\operatorname{cosec}(kL)$ into (*) gives

$$y = \frac{M}{P}\cos(kx) + \left[\frac{M}{P}\tan\left(\frac{kL}{2}\right) + \frac{FL}{P}\operatorname{cosec}(kL)\right]\sin(kx) - \frac{1}{P}(Fx + M)$$

Multiplying by P gives the required result

$$Py = M\cos(kx) + \left[M\tan\left(\frac{kL}{2}\right) + FL\operatorname{cosec}(kL)\right]\sin(kx) - Fx - M$$

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17. Dividing by EI gives the differential equation

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{q}{2EI}\left(x^2 - xL\right) - \frac{M}{EI} \qquad (*)$$

Complementary function y_c;

$$y_c = A\cos(kx) + B\sin(kx) \qquad \text{where} \qquad k = \sqrt{P/EI}$$

Particular integral Y; Since we have a quadratic so our trial function is

$$Y = ax^2 + bx + c \qquad (\dagger)$$

$$\frac{dY}{dx} = 2ax + b, \quad \frac{d^2Y}{dx^2} = 2a$$

Substituting into $\frac{d^2Y}{dx^2} + \frac{P}{EI}Y = \frac{q}{2EI}\left(x^2 - xL\right) - \frac{M}{EI}$ gives

$$2a + \frac{P}{EI}\left(ax^2 + bx + c\right) = \frac{q}{2EI}\left(x^2 - xL\right) - \frac{M}{EI}$$

Equating coefficients of

x^2: $\quad \frac{Pa}{EI} = \frac{q}{2EI}$ gives $a = \frac{q}{2P}$

x: $\quad \frac{Pb}{EI} = -\frac{qL}{2EI}$ gives $b = -\frac{qL}{2P}$

constants: $\quad 2a + \frac{Pc}{EI} = -\frac{M}{EI}$

$$\frac{Pc}{EI} = -\frac{M}{EI} - 2a = -\frac{M}{EI} - \frac{2q}{2P} \quad \text{gives} \quad c = -\frac{M}{P} - \frac{qEI}{P^2}$$

Substituting $a=\frac{q}{2P}$, $b=-\frac{qL}{2P}$ and $c=-\frac{M}{P}-\frac{qEI}{P^2}$ into (†) gives the particular integral

$$Y=\frac{q}{2P}x^2-\frac{qL}{2P}x-\frac{M}{P}-\frac{qEI}{P^2}$$
$$=\frac{1}{2P}\left(qx^2-qLx-2M-\frac{2qEI}{P}\right)$$
$$Y=\frac{1}{2P}\left(qx^2-qLx-2M-\frac{2q}{k^2}\right) \quad \text{remember } k^2=\frac{P}{EI}$$

The general solution, $y=y_c+Y$, is

$$y=A\cos(kx)+B\sin(kx)+\frac{1}{2P}\left(qx^2-qLx-2M-\frac{2q}{k^2}\right) \qquad (††)$$

Substituting $x=0,\ y=0$ into (††) gives

$$0=A+0+\frac{1}{2P}\left(0-0-2M-\frac{2q}{k^2}\right),\ 0=A-\frac{1}{P}\left(M+\frac{q}{k^2}\right), \text{ hence } A=\frac{1}{P}\left(\frac{q}{k^2}+M\right)$$

Differentiating (††) and placing $x=0$, $\frac{dy}{dx}=0$ into the result gives

$$\frac{dy}{dx}=-kA\sin(kx)+kB\cos(kx)+\frac{1}{2P}(2qx-qL)$$
$$0=0+kB-\frac{qL}{2P} \quad \text{gives } B=\frac{qL}{2Pk}$$

Substituting $A=\frac{1}{P}\left(\frac{q}{k^2}+M\right)$ and $B=\frac{qL}{2Pk}$ into (††) yields

$$y=\frac{1}{P}\left(\frac{q}{k^2}+M\right)\cos(kx)+\frac{qL}{2Pk}\sin(kx)+\frac{1}{2P}\left(qx^2-qLx-2M-\frac{2q}{k^2}\right)$$
$$y=\frac{1}{2P}\left[2\left(\frac{q}{k^2}+M\right)\cos(kx)+\frac{qL}{k}\sin(kx)+qx^2-qLx-2M-\frac{2q}{k^2}\right]$$

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18. We have

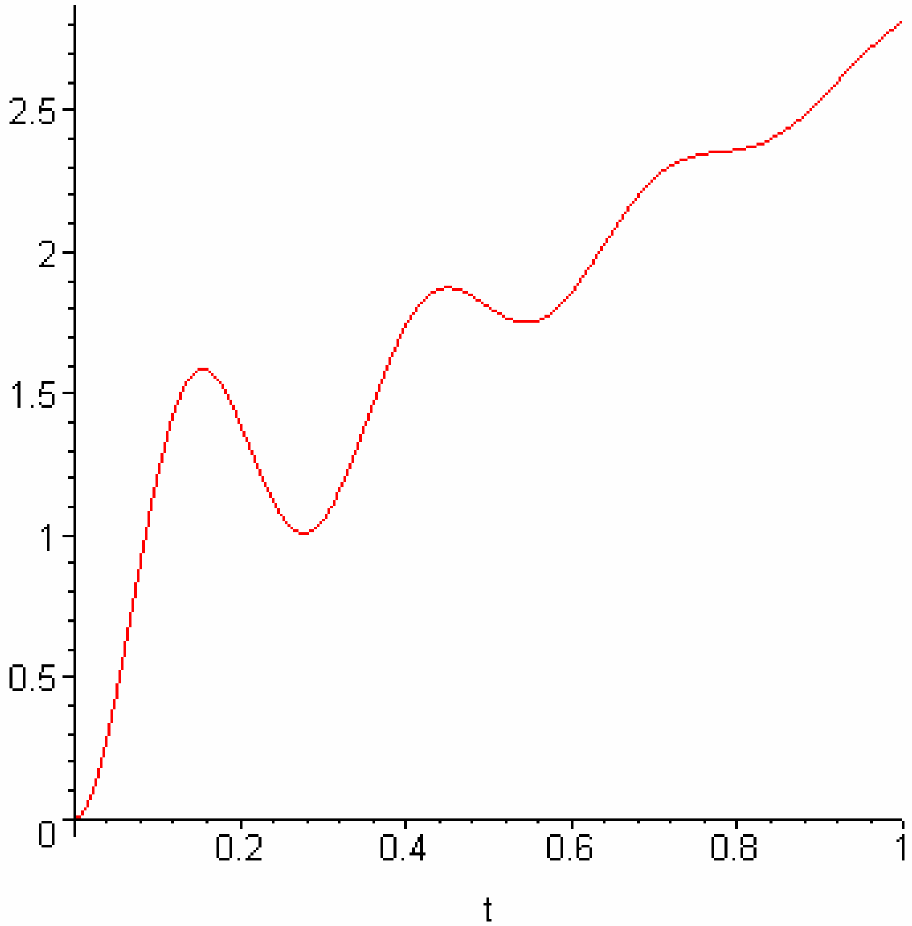
```
> de:=diff(x(t),t,t)+6*diff(x(t),t)+500*x(t)=1000*t+400;
```

$$de:=\left(\frac{d^2}{dt^2}x(t)\right)+6\left(\frac{d}{dt}x(t)\right)+500\,x(t)=1000\,t+400$$

```
> soln:=dsolve({de,x(0)=0,D(x)(0)=0},x(t));
>
```

$$soln:=x(t)=-\frac{541}{61375}e^{(-3\,t)}\sin(\sqrt{491}\,t)\sqrt{491}-\frac{97}{125}e^{(-3\,t)}\cos(\sqrt{491}\,t)+\frac{97}{125}+2\,t$$

```
> plot(rhs(soln),t=0..1);
```


2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1
t