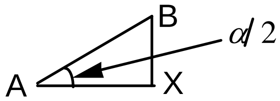
Complete solutions to Miscellaneous Exercise 4

1. Consider one half of the triangle ABE. The point X is midpoint of BE:



BX=$x\sin(\alpha/2)$ and AX=$x\cos(\alpha/2)$.

Area of triangle ABE $=\left[x\cos\left(\frac{\alpha}{2}\right)\right]\times\left[x\sin\left(\frac{\alpha}{2}\right)\right]$

$$= x^2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) \underbrace{=}_{\text{by (4.53)}} \frac{x^2\sin(\alpha)}{2}$$

Area of rectangle BCDE $= y\times\left[2x\sin\left(\frac{\alpha}{2}\right)\right] = 2xy\sin\left(\frac{\alpha}{2}\right)$

Total Area $=\frac{x^2\sin(\alpha)}{2}+2xy\sin\left(\frac{\alpha}{2}\right)=\frac{x}{2}\left[x\sin(\alpha)+4y\sin\left(\frac{\alpha}{2}\right)\right]$

2. By (4.43): $\sin\left(\omega t+\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{2}-\omega t-\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{6}-\omega t\right)=\cos\left[-\left(\omega t-\frac{\pi}{6}\right)\right]$

Using (4.51) we have $\cos\left[-\left(\omega t-\frac{\pi}{6}\right)\right]=\cos\left(\omega t-\frac{\pi}{6}\right)$. So

$$i_1+i_2=\cos\left(\omega t-\frac{\pi}{6}\right)+\cos\left(\omega t-\frac{\pi}{6}\right)=2\cos\left(\omega t-\frac{\pi}{6}\right)$$

3.
$$f_1(t)f_2(t)=A_1A_2\sin(\omega t).\sin(\omega t+\omega\tau+\phi)$$
$$\underbrace{=}_{\text{by (4.59)}}\frac{A_1A_2}{2}\left[\cos(\omega t-\omega t-\omega\tau-\phi)-\cos(\omega t+\omega t+\omega\tau+\phi)\right]$$
$$=\frac{A_1A_2}{2}\left[\cos[-(\omega\tau+\phi)]-\cos(2\omega t+\omega\tau+\phi)\right]$$
$$=\frac{A_1A_2}{2}\left[\cos(\omega\tau+\phi)-\cos(2\omega t+\omega\tau+\phi)\right]$$

4. $v^2=V^2\left[\cos\left(\omega t+\frac{\pi}{4}\right)\right]^2\underbrace{=}_{\text{by (4.39)}}V^2\left[\cos(\omega t)\cos\left(\frac{\pi}{4}\right)-\sin(\omega t)\sin\left(\frac{\pi}{4}\right)\right]^2$

$$=V^2\left[\cos(\omega t)\frac{1}{\sqrt{2}}-\sin(\omega t)\frac{1}{\sqrt{2}}\right]^2$$
$$=\frac{V^2}{2}\left[\cos(\omega t)-\sin(\omega t)\right]^2 \qquad \left(\text{Factorizing } 1/\sqrt{2} \text{ and Squaring}\right)$$
$$=\frac{V^2}{2}\left[\cos^2(\omega t)-2\cos(\omega t)\sin(\omega t)+\sin^2(\omega t)\right]$$
$$v^2=\frac{V^2}{2}\left[1-\sin(2\omega t)\right]$$

(4.39)	$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$
(4.43)	$\sin(A)=\cos(90°-A)$
(4.51)	$\cos(-A)=\cos(A)$
(4.53)	$2\sin(A)\cos(A)=\sin(2A)$
(4.59)	$2\sin(A)\sin(B)=\cos(A-B)-\cos(A+B)$

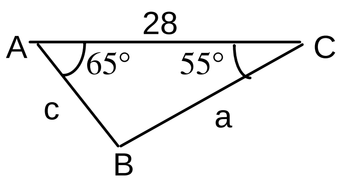
Hence substituting for v^2 gives the result: $P = \frac{v^2}{R} = \frac{V^2}{2R}\left[1 - \sin(2\omega t)\right]$

5.
$$P = 4\sin\left(\omega t + \frac{\pi}{4}\right)\sin(\omega t)$$
$$= 4.\underbrace{\left[\sin(\omega t)\cos\left(\frac{\pi}{4}\right) + \cos(\omega t)\sin\left(\frac{\pi}{4}\right)\right]}_{\text{by }(4.37)}\sin(\omega t)$$
$$= \frac{4}{\sqrt{2}}\left[\sin(\omega t) + \cos(\omega t)\right]\sin(\omega t)$$
$$= \frac{4}{\sqrt{2}}\left[\sin^2(\omega t) + \frac{\sin(2\omega t)}{2}\right]$$
$$= \frac{2}{\sqrt{2}}\left[2\sin^2(\omega t) + \sin(2\omega t)\right]$$
$$P = \sqrt{2}\left[2\sin^2(\omega t) + \sin(2\omega t)\right]$$

6. From the first equation we have $t = \frac{x}{u\cos(\alpha)}$. Substituting this into

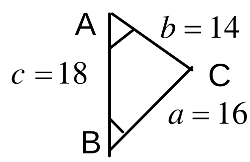
$y = ut\,\sin(\alpha) - \frac{gt^2}{2}$ gives
$$y = \left[\not{u}.\frac{x}{\not{u}\cos(\alpha)}.\sin(\alpha)\right] - \left[\frac{g\,x^2}{2u^2\cos^2(\alpha)}\right] = x\underbrace{\tan(\alpha)}_{\text{by }(4.35)} - \frac{gx^2}{2u^2}\underbrace{\sec^2(\alpha)}_{\text{by }(4.11)}$$
$$y = \frac{1}{2u^2}\left[2u^2x\tan(\alpha) - gx^2\sec^2(\alpha)\right]$$

7. Labelling the triangle according to the sine rule we have



Need to find *a* and *c*:
$$\frac{a}{\sin(65°)} = \frac{c}{\sin(55°)} = \frac{28}{\sin(60°)} = 32.33$$
$$a = 32.33 \times \sin(65°) = 29.30 \text{ and } c = 32.33 \times \sin(55°) = 26.48$$

8.



Need to find angle A and angle B. Using cosine rules:

(4.11) $\sec(A) = 1/\cos(A)$
(4.35) $\sin(A)/\cos(A) = \tan(A)$
(4.37) $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$$\cos(A) = \frac{c^2+b^2-a^2}{2bc} = \frac{18^2+14^2-16^2}{2\times 14\times 18} = 0.524$$

$$\cos(B) = \frac{c^2+a^2-b^2}{2ca} = \frac{18^2+16^2-14^2}{2\times 18\times 16} = 0.667$$

Taking inverse cosine gives: $A = 58.40°$, $B = 48.16°$. How do we find angle C?

$$C = 180° - (58.4° + 48.16°) = 73.44°$$

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9. (a) Dividing $\sin^2(\theta)+\cos^2(\theta)=1$ by $\cos^2(\theta)$ gives (same as **EXAMPLE 30**):

$$\frac{\sin^2(\theta)}{\cos^2(\theta)}+1 = \frac{1}{\cos^2(\theta)}$$

$$\underbrace{\tan^2(\theta)}_{\text{by }(4.35)}+1 = \underbrace{\sec^2(\theta)}_{\text{by }(4.11)}$$

(b) Dividing $\sin^2(\theta)+\cos^2(\theta)=1$ by $\sin^2(\theta)$:

$$1+\frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1+\underbrace{\cot^2(\theta)}_{\text{by }(4.36)} = \underbrace{\operatorname{cosec}^2(\theta)}_{\text{by }(4.10)}$$

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10. $$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} = \frac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2} = \frac{2(\sqrt{2}-1)}{1-(2-2\sqrt{2}+1)}$$

$$= \frac{2(\sqrt{2}-1)}{2\sqrt{2}-2} = \frac{2(\sqrt{2}-1)}{2(\sqrt{2}-1)} = 1$$

We have $\tan(2x)=1$. Taking inverse tan : $2x = 45°$ gives $x = 22.5°$.

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11. (a) $\cos(2x) = 0$ gives $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ (by graph of Fig 53).

Dividing both sides by 2: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(4.10) $$\frac{1}{\sin(A)} = \operatorname{cosec}(A)$$

(4.11) $$\frac{1}{\cos(A)} = \sec(A)$$

(4.35) $$\frac{\sin(A)}{\cos(A)} = \tan(A)$$

(4.36) $$\cot(A) = \frac{\cos(A)}{\sin(A)}$$

(b) $\cos\left(x+\frac{\pi}{6}\right) \underbrace{=}_{\text{by }(4.39)} \cos(x)\cos\left(\frac{\pi}{6}\right)-\sin(x)\sin\left(\frac{\pi}{6}\right)$

$$=\frac{\sqrt{3}}{2}\cos(x)-\frac{1}{2}\sin(x) \text{ (by using TABLE 1)}$$

$$\frac{\sqrt{3}}{2}\cos(x)-\frac{1}{2}\sin(x)=\sin(x)$$

$$\frac{\sqrt{3}}{2}\cos(x)=\frac{3}{2}\sin(x)$$

$$\frac{\sqrt{3}}{3}=\frac{\sin(x)}{\cos(x)} \underbrace{=}_{\text{by }(4.35)} \tan(x)$$

$$\tan(x)=\frac{1}{\sqrt{3}} \quad \text{gives} \quad x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$x=\frac{\pi}{6},\ \frac{7\pi}{6}$$

(c) $\tan(2x).\tan(x)=1$. Applying (4.55) to $\tan(2x)$ gives: $\left[\frac{2\tan(x)}{1-\tan^2(x)}\right]\cdot\tan(x)=1$

$$2\tan^2(x)=1-\tan^2(x)$$

$$3\tan^2(x)=1$$

$$\tan(x)=\pm\frac{1}{\sqrt{3}}$$

$\tan(x)=\frac{1}{\sqrt{3}}$ gives $x=\frac{\pi}{6},\ \frac{7\pi}{6}$ and $\tan(x)=-\frac{1}{\sqrt{3}}$ gives $x=\frac{5\pi}{6},\ \frac{11\pi}{6}$

$x=\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6}$ and $\frac{11\pi}{6}$

(d) We have $4\sin(x)\cos(x)-4\cos^2(x)=-2$. Dividing by 2:

$$2\sin(x)\cos(x)-2\cos^2(x)=-1$$

$$2\sin(x)\cos(x)=2\cos^2(x)-1$$

$$\underbrace{\sin(2x)}_{\text{by }(4.53)}=\underbrace{\cos(2x)}_{\text{by }(4.54)}$$

Dividing through by $\cos(2x)$; $\tan(2x)=1$ gives $2x=\pi/4,\ 5\pi/4,\ 9\pi/4,\ 13\pi/4$.
Hence $x=\frac{\pi}{8},\ \frac{5\pi}{8},\ \frac{9\pi}{8},\ \frac{13\pi}{8}$.

(4.35) $\frac{\sin(A)}{\cos(A)}=\tan(A)$

(4.39) $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$

(4.53) $2\sin(A)\cos(A)=\sin(2A)$

(4.54) $2\cos^2(A)-1=\cos(2A)$

(4.55) $\tan(2A)=\frac{2\tan(A)}{1-\tan^2(A)}$

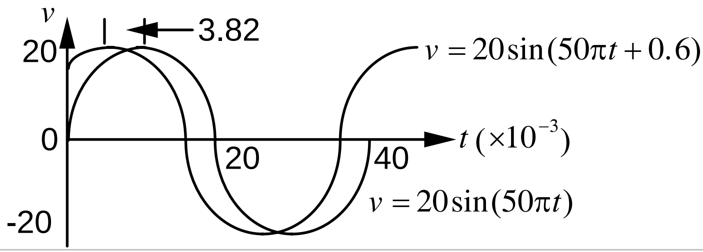
12. (i) Amplitude =20V, period $T = \dfrac{2\pi}{50\pi} = \dfrac{1}{25}$ and frequency 25.

(ii) time displacement $= \dfrac{0.6}{50\pi} = 3.82 \times 10^{-3}$.

(iii) $20\sin(50\pi t + 0.6)$ leads $20\sin(50\pi t)$ by 3.82×10^{-3}.

$20\sin(50\pi t)$ is zero when $\sin(50\pi t) = 0$. Thus $50\pi t = 0,\ \pi,\ 2\pi$:

$$t = 0,\ \frac{1}{50},\ \frac{1}{25} = 0,\ 20,\ 40\ \left(\times 10^{-3}\right)$$



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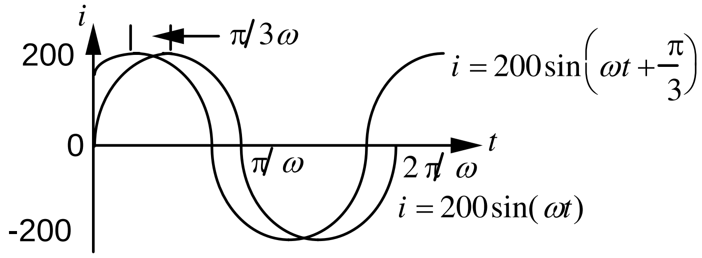
13. We have $200\sin\left(\omega t + \dfrac{\pi}{3}\right)$ leads $200\sin(\omega t)$ by $\dfrac{\pi}{3\omega}$ *radians*. The $200\sin(\omega t)$ graph is similar to the $\sin(t)$ graph shown in Fig 52 but it cuts the □ axis at $\dfrac{\pi}{\omega},\ \dfrac{2\pi}{\omega},\ \dfrac{3\pi}{\omega}$ and has an amplitude of 200:



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14. $x_1 = 0$ when $\sin\left(\dfrac{\pi t}{5}\right) = 0$ gives $\dfrac{\pi t}{5} = \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi, \ldots$ Multiplying by $5/\pi$:

$$t = 5,\ 10,\ 15,\ 20,\ 25, \ldots$$

Similarly $x_2 = 0$ when $\sin\left(\dfrac{\pi t}{4}\right) = 0$ gives $t = 4,\ 8,\ 16,\ 20, \ldots$

Meet at $x_1 = x_2 = 0$ after □=20s.

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15. We have $x_1 = x_2$ gives $0.3\sin\left(\dfrac{\pi t}{3}\right) = 0.3\cos\left(\dfrac{\pi t}{3}\right)$. Dividing both sides by 0.3:

$$\sin\left(\frac{\pi t}{3}\right) = \cos\left(\frac{\pi t}{3}\right)$$

Dividing both sides by $\cos\left(\dfrac{\pi t}{3}\right)$:

By (4.35) $\tan\left(\dfrac{\pi t}{3}\right) = 1$ gives $\dfrac{\pi t}{3} = \dfrac{\pi}{4}$. Thus $t = \dfrac{3}{4}$.

(4.35) $$\frac{\sin(A)}{\cos(A)} = \tan(A)$$

16. Substituting $x=0,\ y=e$ into $y=A\cos(kx)+B\sin(kx)$ gives:

$$e=A$$

Substituting $x=L,\ y=e$ gives:

$$e=e\cos(kL)+B\sin(kL)$$

$$B=\frac{e[1-\cos(kL)]}{\sin(kL)} \quad \text{gives} \quad B=e\tan\left(\frac{kL}{2}\right) \qquad \text{(by using hint)}$$

Substituting $A=e$ and $B=e\tan\left(\frac{kL}{2}\right)$ into $y=A\cos(kx)+B\sin(kx)$ gives

$$y=e\left[\cos(kx)+\tan\left(\frac{kL}{2}\right)\sin(kx)\right]$$

At $x=\frac{L}{2}$: $y=e\left[\cos\left(\frac{kL}{2}\right)+\tan\left(\frac{kL}{2}\right)\sin\left(\frac{kL}{2}\right)\right]$

$$=e\left(\frac{\cos^2\left(\frac{kL}{2}\right)+\sin^2\left(\frac{kL}{2}\right)}{\cos\left(\frac{kL}{2}\right)}\right)\underbrace{=}_{\text{by }(4.64)}e\left[\frac{1}{\cos\left(\frac{kL}{2}\right)}\right]=\underbrace{e\sec\left(\frac{kL}{2}\right)}_{\text{by }(4.11)}$$

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17. (i) $\cos(x)-\sec(x)=1$. By (4.11):

$$\cos(x)-\frac{1}{\cos(x)}=1$$

$$\cos^2(x)-1=\cos(x)$$

$$\cos^2(x)-\cos(x)-1=0$$

Let $y=\cos(x)$ then we have:

$$y^2-y-1=0$$

$$y=\frac{1\pm\sqrt{1+4}}{2}=\frac{1}{2}\pm\sqrt{\frac{5}{2}}=1.618,\ -0.618$$

No solution for $\cos(x)=1.618$ because the cosine function lies between -1 and 1. The solutions for $\cos(x)=-0.618$ are $128.17°$ and $231.83°$.

(ii) $\cos(x)+\sec(x)=1$. By (4.11):

$$\cos(x)+\frac{1}{\cos(x)}=1$$

$$\cos^2(x)+1=\cos(x)$$

$$\cos^2(x)-\cos(x)+1=0$$

Similar to (i): $\cos(x)=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{-3}}{2}$. Since $\sqrt{-3}$ is not a real number, there is **no** real solution to the equation $\cos(x)+\sec(x)=1$.

(iii) We have

$$\cos^2(x)=2\cos(x)+\tan^2(x)$$

$$\cos^2(x)-2\cos(x)-\tan^2(x)=0$$

(4.11) $\qquad \frac{1}{\cos(A)}=\sec(A)$

(4.64) $\qquad \sin^2(A)+\cos^2(A)=1$

Let $y=\cos(x)$, so we have $y^2-2y-\tan^2(x)=0$:

$$y=\frac{2\pm\sqrt{4+4\tan^2(x)}}{2}=\frac{2}{2}\pm\frac{\sqrt{4+4\tan^2(x)}}{2}$$
$$=1\pm\frac{\sqrt{4}}{2}\sqrt{1+\tan^2(x)}$$
$$=1\pm\sqrt{1+\tan^2(x)}$$
$$=1\pm\sqrt{\underbrace{\sec^2(x)}_{\text{by (4.65)}}}$$
$$y=1\pm\sec(x)$$

Putting $y=\cos(x)$ gives:

$$\cos(x)=1+\sec(x) \quad \text{or} \quad \cos(x)=1-\sec(x)$$
$$\cos(x)-\sec(x)=1 \quad \text{or} \quad \cos(x)+\sec(x)=1$$

By (i) and (ii) the only solutions are $128.17°$ and $231.83°$ to the equation: $\cos^2(x)=2\cos(x)+\tan^2(x)$.

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18. (a) $\dfrac{\tan(40°)+\tan(20°)}{1-\tan(40°)\tan(20°)}\underbrace{=}_{\text{by (4.41)}}\tan(40°+20°)=\tan(60°)\underbrace{=}_{\text{by TABLE 1}}\sqrt{3}$

(b) $\cos(15°)-\sin(15°)\underbrace{=}_{\text{by (4.75)}}\sqrt{2}\cos(15°-(-45°))$

$$=\sqrt{2}\cos(60°)=\sqrt{2}\left(\frac{1}{2}\right)=\frac{1}{\sqrt{2}}$$

(c) $\cos(75°)+\sin(75°)\underbrace{=}_{\text{by (4.75)}}\sqrt{2}\cos(75°-45°)=\sqrt{2}\cos(30°)=\sqrt{2}\cdot\dfrac{\sqrt{3}}{2}=\sqrt{\dfrac{3}{2}}$

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19. We have

$$\tan(2x)+\tan(x)=\frac{2\tan(x)}{1-\tan^2(x)}+\tan(x)=0$$
$$2\tan(x)+\tan(x)-\tan^3(x)=0$$
$$3\tan(x)-\tan^3(x)=0$$
$$\tan(x)\left(3-\tan^2(x)\right)=0$$
$$\tan(x)=0 \quad \text{or} \quad \tan(x)=\pm\sqrt{3}$$
$$x=0,\ \pi,\ 2\pi,\ldots \qquad x=\pi/3,\ 2\pi/3,\ 4\pi/3,\ 5\pi/3\ldots$$
$$x=n\pi/3 \quad (\text{where } n \text{ is a whole number})$$

(4.41) $\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}=\tan(A+B)$

(4.51) $\cos(-A)=\cos(A)$

(4.65) $1+\tan^2(A)=\sec^2(A)$

(4.75) $a\cos(\theta)+b\sin(\theta)=R\cos(\theta-\beta)$ with $R=\sqrt{a^2+b^2}$ and $\beta=\tan^{-1}(b/a)$