Complete solutions to Miscellaneous Exercise 6

1. If i = constant then $\frac{di}{dt} = 0$. Hence $v = 0$.

2. $$v = 3e^{2t}, \quad \frac{dv}{dt} = (2\times 3)e^{2t} = 6e^{2t}$$

Substituting:

$$\frac{dv}{dt} - 2v = 6e^{2t} - 2\left(3e^{2t}\right) = 6e^{2t} - 6e^{2t} = 0$$

3. We have $v = ae^{bt}$, differentiating gives:

$$\frac{dv}{dt} = bae^{bt}$$

Hence

$$\frac{dv}{dt} - bv = bae^{bt} - b.ae^{bt} = 0$$

4. Substituting $T = -10x^3 - 500x + 600$:

$$Q = -kA\frac{d}{dx}\left(-10x^3 - 500x + 600\right)$$

$$= -kA\left(-30x^2 - 500\right) = kA\left(30x^2 + 500\right) \quad \text{[Taking out a minus]}$$

Substituting $x = t$, $k = 0.7$ and $A = 10$ into Q gives:

$$Q = 7\left(30t^2 + 500\right) = 210t^2 + 3500 \quad \text{[Multiplying by 7]}$$

5. Differentiating $T = 100e^{-3x}$ gives

$$\frac{dT}{dx} = (-3)\times 100e^{-3x} = -300e^{-3x}$$

Substituting $k = 370$ and $A = 0.01\pi$ into Q gives

$$Q = -370 \times 0.01\pi \times\left(-300e^{-3x}\right) = 3487.17e^{-3x}$$

At $x = t$, $Q = \left(3.49\times 10^3\right)e^{-3t}$

6. Differentiating with respect to θ:

$$\frac{dV}{d\theta} = -kl^2\cos(\theta)\sin(\theta) + mgl\cos(\theta)$$

$$\underbrace{=}_{\text{factorizing}} l\cos(\theta)\left[mg - kl\sin(\theta)\right] = 0 \qquad \text{(for equilibrium)}$$

$$\cos(\theta) = 0 \text{ gives } \theta = \pi/2$$

$$\text{or } mg - kl\sin(\theta) = 0 \text{ gives } \sin(\theta) = \frac{mg}{kl} \text{ hence } \theta = \sin^{-1}\left(\frac{mg}{kl}\right).$$

7. $$\frac{dP}{dx} = kx - mg = 0$$

$$x = mg/k$$

8. Substituting $C = 3\mu = 3\times 10^{-6}$ into $i = C\frac{dv}{dt}$:

$$i = \left(3\times10^{-6}\right)\frac{d}{dt}\left[-6e^{-\left(2\times10^3\right)t} + 10e^{-\left(8\times10^3\right)t}\right]$$

$$\underbrace{=}_{\text{by (6.11)}} 3\times10^{-6}\left[\left[\left(-2\times10^3\right)\times(-6)\right]e^{-\left(2\times10^3\right)t} - \left(8\times10^3\times10\right)e^{-\left(8\times10^3\right)t}\right]$$

$$= 0.036e^{-\left(2\times10^3\right)t} - 0.24e^{-\left(8\times10^3\right)t} \quad [\text{Simplifying}]$$

9. Substituting $R(t) = e^{e^t-1}$ gives $\frac{d}{dt}\left[1-e^{e^t-1}\right] = -\frac{d}{dt}\left[e^{e^t-1}\right]$

Let $u = e^t - 1$, $\frac{du}{dt} = e^t$

By (6.16) with $u = e^{t-1}$ we have $-\frac{d}{dt}\left[e^{e^t-1}\right] = -e^{e^t-1}.e^t = -e^{e^t+t-1}$

10. With $v = 50\sin(75t)$ we have

$$\frac{dv}{dt} \underbrace{=}_{\text{by (6.12)}} 75\times50\cos(75t) = 3750\cos(75t)$$

$$i = \left(10\times10^{-6}\right)\times3750\cos(75t) = 0.0375\cos(75t)\ amps$$

11. Differentiating M gives

$$V = \frac{dM}{dx} = 0 - 0 + \omega_0 L + \frac{kL^4}{4} - \frac{2\,\omega_0 x}{2} - \frac{5kx^4}{20}$$

$$= \omega_0 L + \frac{kL^4}{4} - \omega_0 x - \frac{kx^4}{4}$$

$$= \omega_0(L-x) + \frac{k}{4}\left(L^4 - x^4\right) \quad [\text{Factorizing}]$$

$$= \omega_0(L-x) + \frac{k}{4}(L-x)\left(L^3 + L^2x + Lx^2 + x^3\right) \quad [\text{Using hint}]$$

$$= (L-x)\left[\omega_0 + \frac{k}{4}\left(L^3 + L^2x + Lx^2 + x^3\right)\right] \quad [\text{Factorizing}]$$

12.

$$V = \frac{d}{dx}\left\{\frac{\omega_0 L}{\pi}\left[L + x + \frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\right]\right\}$$

$$= \frac{\omega_0 L}{\pi}\left[0 + 1 + \underbrace{\frac{L}{\pi}.\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)}_{\text{by (6.12)}}\right] = \frac{\omega_0 L}{\pi}\left[1 + \cos\left(\frac{\pi x}{L}\right)\right] \quad \left(\text{Cancelling } \frac{L}{\pi}.\frac{\pi}{L} = 1\right)$$

(6.11) $\left(e^{kt}\right)' = ke^{kt}$

(6.12) $[\sin(kx)]' = k\cos(kx)$

(6.16) $\frac{d}{dt}\left[e^u\right] = e^u\frac{du}{dt}$

13. By (6.16) with $x = e^{(-2+\sqrt{3})\omega t} + e^{-(2+\sqrt{3})\omega t}$ we have

$$\dot{x} = \left(-2+\sqrt{3}\right)\omega e^{\left(-2+\sqrt{3}\right)\omega t} - \left(2+\sqrt{3}\right)\omega e^{-\left(2+\sqrt{3}\right)\omega t}$$

$$\ddot{x} = \left(-2+\sqrt{3}\right)^2 \omega^2 e^{\left(-2+\sqrt{3}\right)\omega t} + \left(2+\sqrt{3}\right)^2 \omega^2 e^{-\left(2+\sqrt{3}\right)\omega t}$$

Substituting into $\ddot{x} + 4\omega\dot{x} + \omega^2 x$ gives

$$\left(-2+\sqrt{3}\right)^2 \omega^2 e^{\left(-2+\sqrt{3}\right)\omega t} + \left(2+\sqrt{3}\right)^2 \omega^2 e^{-\left(2+\sqrt{3}\right)\omega t}$$

$$+4\omega\left[\left(-2+\sqrt{3}\right)\omega e^{\left(-2+\sqrt{3}\right)\omega t} - \left(2+\sqrt{3}\right)\omega e^{-\left(2+\sqrt{3}\right)\omega t}\right] + \omega^2\left[e^{\left(-2+\sqrt{3}\right)\omega t} + e^{-\left(2+\sqrt{3}\right)\omega t}\right]$$

$$= \underbrace{\left[\left(-2+\sqrt{3}\right)^2\omega^2 + 4\omega\left(-2+\sqrt{3}\right)\omega + \omega^2\right]}_{\text{collecting the terms of } e^{\left(-2+\sqrt{3}\right)\omega t}} e^{\left(-2+\sqrt{3}\right)\omega t}$$

$$+\underbrace{\left[\left(2+\sqrt{3}\right)^2\omega^2 - 4\omega\left(2+\sqrt{3}\right)\omega + \omega^2\right]}_{\text{collecting the terms of } e^{-\left(2+\sqrt{3}\right)\omega t}} e^{-\left(2+\sqrt{3}\right)\omega t}$$

$$= \underbrace{\left[\left(4-4\sqrt{3}+3\right)-8+4\sqrt{3}+1\right]}_{\text{expanding the brackets and taking out a factor of } \omega^2}\omega^2 e^{\left(-2+\sqrt{3}\right)\omega t} + \underbrace{\left[4+4\sqrt{3}+3-8-4\sqrt{3}+1\right]}_{=0}\omega^2 e^{-\left(2+\sqrt{3}\right)\omega t}$$

$$= 0 + 0 = 0$$

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14. We can rewrite v as $v = A\left(1+e^{-kx}\right)^{-1}$, differentiating this by using (6.14) gives:

$$E = -\frac{dv}{dx} = -(-1)A\left(1+e^{-kx}\right)^{-2}\left(-ke^{-kx}\right) \quad [\text{By } (6.14)]$$

$$= \frac{-kA}{e^{kx}\left(1+e^{-kx}\right)^2} = \frac{-kA}{e^{kx}\left(1+2e^{-kx}+e^{-2kx}\right)} \quad [\text{Expanding Denominator}]$$

$$= \frac{-kA}{e^{kx}+2+e^{-kx}}$$

$$= \frac{-kA}{2\left[1+\left(e^{kx}+e^{-kx}\right)/2\right]}$$

$$\underbrace{=}_{\text{by } (5.24)} \frac{-kA}{2\left[1+\cosh(kx)\right]}$$

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15. By using the product rule (6.31) with $u = t^2$ and $v = \sin(t)$:

$$v_x = 2t\sin(t) + t^2\cos(t)$$

$$a_x \underbrace{=}_{\text{by } (6.31)} \left[2\sin(t) + 2t\cos(t)\right] + \left[2t\cos(t) - t^2\sin(t)\right]$$

$$= 2\sin(t) + 4t\cos(t) - t^2\sin(t)$$

(5.24) $\left(e^x + e^{-x}\right)/2 = \cosh(x)$

(6.14) $\frac{d}{dt}\left(u^n\right) = nu^{n-1}\frac{du}{dt}$

(6.16) $\frac{d}{dt}\left[e^u\right] = e^u\frac{du}{dt}$

16. (i) We use the product rule (6.31) on $x = t\cos(t)$

$$v_x = \dot{x} = \cos(t) - t\sin(t)$$
$$a_x = \ddot{x} = -\sin(t) - [\sin(t) + t\cos(t)]$$
$$a_x = -2\sin(t) - t\cos(t)$$

We apply (6.31) on $y = t\sin(t)$

$$v_y = \dot{y} = \sin(t) + t\cos(t)$$
$$a_y = \ddot{y} = \cos(t) + [\cos(t) - t\sin(t)] = 2\cos(t) - t\sin(t)$$

(ii) By using the given definitions:

$$v = \left\{v_x{}^2 + v_y{}^2\right\}^{1/2}$$
$$= \left\{[\cos(t) - t\sin(t)]^2 + [\sin(t) + t\cos(t)]^2\right\}^{1/2} \quad [\text{Substituting for } v_x \text{ and } v_y]$$
$$= \left\{[\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t)] + [\sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t)]\right\}^{1/2}$$
$$= \left\{\underbrace{\cos^2(t) + \sin^2(t)}_{=1} + t^2\underbrace{[\sin^2(t) + \cos^2(t)]}_{=1}\right\}^{1/2}$$
$$v = (1 + t^2)^{1/2} = \sqrt{1 + t^2}$$

Now we need to show the result for a.

$$a = \left\{a_x{}^2 + a_y{}^2\right\}^{1/2}$$
$$= \left\{[-2\sin(t) - t\cos(t)]^2 + [2\cos(t) - t\sin(t)]^2\right\}^{1/2}$$
$$= \left\{[2\sin(t) + t\cos(t)]^2 + [2\cos(t) - t\sin(t)]^2\right\}^{1/2}$$
$$= \left\{\begin{matrix}[4\sin^2(t) + 4t\sin(t)\cos(t) + t^2\cos^2(t)] \\ +[4\cos^2(t) - 4t\cos(t)\sin(t) + t^2\sin^2(t)]\end{matrix}\right\}^{1/2} \quad [\text{Expanding}]$$
$$= \left\{4\underbrace{[\sin^2(t) + \cos^2(t)]}_{=1} + t^2\underbrace{[\cos^2(t) + \sin^2(t)]}_{=1}\right\}^{1/2}$$
$$a = (4 + t^2)^{1/2} = \sqrt{4 + t^2}$$

(iii) $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{v_y}{v_x} = \dfrac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$

17. Very similar to the solution of **EXERCISE 6(g)**, question 4(d) with e replaced by 10.

$$\frac{dy}{dx} = 10^{(x^x)}x^x[\ln(x) + 1]\ln(10)$$

(6.31) $\quad (uv)' = u'v + uv'$