Complete solutions to Miscellaneous Exercise 8

1. We have

$$KE = m\int_0^v v\,dv = m\left[\frac{v^2}{2}\right]_0^v = \frac{mv^2}{2}$$

2. We have

$$W = \int_0^l \frac{EAx}{L}\,dx = \left[\frac{EAx^2}{2L}\right]_0^l = \frac{EAl^2}{2L}$$

3. We have

$$W = k\int_{x_1}^{x_2} x\,dx = k\left[\frac{x^2}{2}\right]_{x_1}^{x_2} = \frac{1}{2}k\left(x_2^2 - x_1^2\right)$$

4. Multiplying both sides by μ gives

$$\mu\theta = \int_{T_1}^{T_2} \frac{dT}{T} \underbrace{=}_{\text{by (8.2)}} \left[\ln|T|\right]_{T_1}^{T_2} = \ln(T_2) - \ln(T_1) \underbrace{=}_{\text{by (5.12)}} \ln\left(\frac{T_2}{T_1}\right)$$

Taking exponentials of both sides:

$$e^{\mu\theta} = \left(\frac{T_2}{T_1}\right) \qquad \left(\text{Because } e^{\ln(x)} = x\right)$$

Hence

$$T_2 = T_1 e^{\mu\theta} \qquad \left(\text{Multiplying by } T_1\right)$$

5.

$$W = \frac{pq}{6EI}\int_0^L \left(2L^3x - 3L^2x^2 + x^4\right)dx$$

$$= \frac{pq}{6EI}\left[\frac{2L^3x^2}{2} - \frac{3L^2x^3}{3} + \frac{x^5}{5}\right]_0^L = \frac{pq}{6EI}\left[L^5 - L^5 + \frac{L^5}{5}\right] = \frac{pqL^5}{30EI}$$

6. Taking out the 2π gives

$$J = 2\pi\int_{D_i/2}^{D_0/2} r^3\,dr$$

$$= 2\pi\left[\frac{r^4}{4}\right]_{D_i/2}^{D_0/2}$$

$$= \frac{\pi}{2}\left[\left(\frac{D_0}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4\right]$$

$$= \frac{\pi}{2}\left[\left(\frac{D_0{}^4}{16} - \frac{D_i{}^4}{16}\right)\right] = \frac{\pi}{32}\left[D_0{}^4 - D_i{}^4\right]$$

7. We first write the integrand into partial fractions and then integrate.
By **EXAMPLE 27** we have the identity

$$\frac{x^3}{x^2-4} = x + 2\left[\frac{1}{x-2} + \frac{1}{x+2}\right]$$

Hence

(5.12) $\qquad \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$

$$\int \frac{x^3}{x^2-4}dx = \int x + 2\left[\int \frac{dx}{x-2} + \int \frac{dx}{x+2}\right]$$

$$= \frac{x^2}{2} + 2\left[\ln|x-2| + \ln|x+2|\right] + C$$

$$= \frac{x^2}{2} + 2\underbrace{\left[\ln|(x-2)(x+2)|\right]}_{\text{by (5.11)}} + C = \frac{x^2}{2} + 2\ln|x^2-4| + C = \frac{x^2}{2} + \ln\left(x^2-4\right)^2 + C$$

8. Integrating gives:

$$\Delta h = \left[1.8T + \frac{\left(12\times10^{-3}\right)T^2}{2}\right]_{200}^{1000}$$

$$= \left[(1.8\times1000) + \left(6\times10^{-3}\right)1000^2\right] - \left[(1.8\times200) + \left(6\times10^{-3}\right)200^2\right]$$

$$= 7200 = 7.2 \text{ kJ/kg}$$

9.

$$\Delta h = R\int_{T_1}^{T_2}\left(a + bT + cT^2 + dT^3 + eT^4\right)dT$$

$$= R\left[aT + \frac{b}{2}T^2 + \frac{c}{3}T^3 + \frac{d}{4}T^4 + \frac{e}{5}T^5\right]_{T_1}^{T_2}$$

$$= R\left[a\left(T_2 - T_1\right) + \frac{b}{2}\left(T_2^2 - T_1^2\right) + \frac{c}{3}\left(T_2^3 - T_1^3\right) + \frac{d}{4}\left(T_2^4 - T_1^4\right) + \frac{e}{5}\left(T_2^5 - T_1^5\right)\right]$$

10. We can use the trigonometric identity for $\cos(2x) = \cos^2(x) - \sin^2(x)$:

$$\int \frac{\cos(2x)}{\cos^2(x)\sin^2(x)}dx = \int \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)\sin^2(x)}dx$$

$$= \int \frac{\cos^2(x)}{\cos^2(x)\sin^2(x)}dx - \int \frac{\sin^2(x)}{\cos^2(x)\sin^2(x)}dx$$

$$= \int \frac{dx}{\sin^2(x)} - \int \frac{dx}{\cos^2(x)}$$

$$= \int \operatorname{cosec}^2(x)\,dx - \int \sec^2(x)\,dx = -\cot(x) - \tan(x) + C$$

11.

$$V = -\int_z^r \frac{q}{2\pi\varepsilon_0 r}dr = -\frac{q}{2\pi\varepsilon_0}\int_z^r \frac{dr}{r}$$

$$\underbrace{=}_{\text{by (8.2)}} -\frac{q}{2\pi\varepsilon_0}\left[\ln(r)\right]_z^r = -\frac{q}{2\pi\varepsilon_0}\left[\ln(r) - \ln(z)\right]$$

$$= \frac{q}{2\pi\varepsilon_0}\left[\ln(z) - \ln(r)\right] \underbrace{=}_{\text{by (5.12)}} \frac{q}{2\pi\varepsilon_0}\ln\left(\frac{z}{r}\right)$$

(5.11) $\ln(A) + \ln(B) = \ln(AB)$

(5.12) $\ln(A) - \ln(B) = \ln(A/B)$

(8.2) $\int dr/r = \ln|r|$

12. From $PV = C$ we have $P = \frac{C}{V}$. Substituting this into the integral gives

$$\begin{aligned}
W &= C\int_{V_1}^{V_2} \frac{dV}{V} \\
&= C\left[\ln|V|\right]_{V_1}^{V_2} \\
&= C\left[\ln(V_2) - \ln(V_1)\right] \\
&= C\ln\left(\frac{V_2}{V_1}\right) = \underbrace{PV}_{=C}\ln\left(\frac{V_2}{V_1}\right)
\end{aligned}$$

13. We have $P = \frac{C}{V^{1.32}} = CV^{-1.32}$. Substituting this into W gives:

$$W = \int_{V_1}^{V_2} CV^{-1.32}dV = C\int_{V_1}^{V_2} V^{-1.32}dV$$

$$\underbrace{=}_{\text{by (8.1)}} C\left[\frac{V^{-1.32+1}}{-1.32+1}\right]_{V_1}^{V_2} = C\left[\frac{V^{-0.32}}{-0.32}\right]_{V_1}^{V_2}$$

$$= C\left[\frac{V_2^{-0.32} - V_1^{-0.32}}{-0.32}\right] = \frac{C}{0.32}\left[V_1^{-0.32} - V_2^{-0.32}\right]$$

14.

(i) Area of 100 rectangles

$$\begin{aligned}
&= \left[\frac{1}{100}\times\left(\frac{1}{100}\right)^3\right] + \left[\frac{1}{100}\times\left(\frac{2}{100}\right)^3\right] + \ldots + \left[\frac{1}{100}\times\left(\frac{99}{100}\right)^3\right] \\
&= \left[\frac{1}{100}\times\frac{1^3}{100^3}\right] + \left[\frac{1}{100}\times\frac{2^3}{100^3}\right] + \ldots + \left[\frac{1}{100}\times\frac{99^3}{100^3}\right] \\
&= \frac{1^3}{100^4} + \frac{2^3}{100^4} + \ldots + \frac{99^3}{100^4} \\
&= \frac{1}{100^4}\left[1^3 + 2^3 + \ldots + 99^3\right] = \frac{1}{100^4}\underbrace{\left[\frac{99^2}{4}(99+1)^2\right]}_{\text{by hint}} = 0.245025
\end{aligned}$$

(ii) $\int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} = 0.25$

(iii) Difference $= 0.25 - 0.245025 = 0.004975$. By considering more rectangles.

15. $P = k\int_{-\infty}^{x}\left(x^{-4} + r^3x^{-7}\right)dx = k\left[\frac{x^{-3}}{-3} + \frac{r^3x^{-6}}{-6}\right]_{-\infty}^{x} = -\frac{k}{6}\left[\frac{r^3}{x^6} + \frac{2}{x^3}\right]$

16. (i)

$$P = \int_0^L w_0 \sin\left(\frac{\pi x}{L}\right)dx = w_0\int_0^L \sin\left(\frac{\pi x}{L}\right)dx$$

$$\underbrace{=}_{\text{by (8.39)}} -w_0\frac{\left[\cos\left(\frac{\pi x}{L}\right)\right]_0^L}{\pi/L} = -\frac{w_0L}{\pi}\underbrace{\left[\cos(\pi) - \cos(0)\right]}_{=-2} = \frac{2w_0L}{\pi}$$

(8.39) $\int \sin(kx+m)dx = -\cos(kx+m)/k$

(ii) $R = w_0\int_0^L x\sin\left(\frac{\pi x}{L}\right)dx$. How do we integrate this?

Use integration by parts formula(8.45)

$$u = x \qquad v' = \sin\left(\frac{\pi x}{L}\right)$$

$$u' = 1 \qquad v = \int \sin\left(\frac{\pi x}{L}\right)dx = \frac{-\cos\left(\frac{\pi x}{L}\right)}{\pi/L} = -\frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)$$

$$R = w_0\left\{\left[-\frac{xL}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]_0^L + \frac{L}{\pi}\int_0^L (1)\cos\left(\frac{\pi x}{L}\right)dx\right\}$$

$$= w_0\left\{\left[-\frac{L^2}{\pi}\underbrace{\cos\left(\frac{\pi L}{L}\right)}_{=-1} - 0\right] + \frac{L^2}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\right]_0^L\right\} = w_0\left\{\frac{L^2}{\pi} + 0\right\} = \frac{w_0 L^2}{\pi}$$

17. Integrating gives:

$$t/RC = -\left[\ln(w - v)\right]_{v=-w}^{v=v}$$

$$-t/RC = \left[\ln(w - v) - \ln(2w)\right]$$

$$-t/RC \underbrace{=}_{\text{by (5.12)}} \ln\left(\frac{w-v}{2w}\right)$$

Taking exponential gives:

$$\frac{w-v}{2w} = e^{-t/RC}$$

$$w - v = 2we^{-t/RC}$$

$$v = w\left(1 - 2e^{-t/RC}\right)$$

18. We have:

$$i = \frac{1}{10\times10^{-3}}\int_0^t\left[-6e^{-\left(2\times10^3\right)x} + 10e^{-\left(8\times10^3\right)x}\right]dx$$

$$\underbrace{=}_{\text{by (8.41)}} \frac{1}{10\times10^{-3}}\left[\frac{-6}{-2\times10^3}e^{-\left(2\times10^3\right)x} - \frac{10}{8\times10^3}e^{-\left(8\times10^3\right)x}\right]_0^t$$

$$= \left[0.3e^{-\left(2\times10^3\right)x} - 0.125e^{-\left(8\times10^3\right)x}\right]_0^t$$

$$= 0.3e^{-\left(2\times10^3\right)t} - 0.125e^{-\left(8\times10^3\right)t} - \left(0.3e^0 - 0.125e^0\right)$$

$$i = 0.3e^{-\left(2\times10^3\right)t} - 0.125e^{-\left(8\times10^3\right)t} - 0.175$$

(5.12) $\ln(A) - \ln(B) = \ln(A/B)$

(8.41) $\int e^{kt+m}dt = e^{kt+m}/k$

(8.45) $\int uv'dx = uv - \int(u'v)dx$

19. (i)

$$i=\frac{1}{5\times10^{-3}}\int_0^t 10\sin(100\pi t)dt=\frac{10}{5\times10^{-3}}\left[\frac{-\cos(100\pi t)}{(100\pi)}\right]_0^t$$

$$=\frac{10}{5\times10^{-3}}\frac{\left[-\cos(100\pi t)+1\right]}{100\pi}=\frac{10}{0.5\pi}\left[1-\cos(100\pi t)\right]=\frac{20}{\pi}\left[1-\cos(100\pi t)\right]$$

(ii) Using the formula given

$$w=\frac{1}{2}\left(5\times10^{-3}\right)\left[\frac{20}{\pi}\left[1-\cos(100\pi t)\right]\right]^2$$

$$=\frac{1}{2}\left(5\times10^{-3}\right)\frac{400}{\pi^2}\left[1-\cos(100\pi t)\right]^2$$

$$w=\frac{\left[1-\cos(100\pi t)\right]^2}{\pi^2}$$

20.

$$\Phi=\frac{\mu iNh}{2\pi}\int_a^b\frac{1}{r}dr=\frac{\mu iNh}{2\pi}\underbrace{\left[\ln(r)\right]_a^b}_{\text{by }(8.2)}=\frac{\mu iNh}{2\pi}\left[\ln(b)-\ln(a)\right]=\frac{\mu iNh}{2\pi}\ln\left(\frac{b}{a}\right)$$

Hence

$$L=N\frac{d}{di}\left[\frac{\mu iNh}{2\pi}\ln\left(\frac{b}{a}\right)\right]=N\frac{\mu Nh}{2\pi}\ln\left(\frac{b}{a}\right)=\frac{\mu N^2h}{2\pi}\ln\left(\frac{b}{a}\right)$$

21. We have

$$\int_0^1\frac{4dx}{1+x^2}\underbrace{=}_{\text{by }(8.26)}4\left[\tan^{-1}(x)\right]_0^1=4\left[\tan^{-1}(1)-\tan^{-1}(0)\right]=4\left[\frac{\pi}{4}-0\right]=\pi$$

22.

$$I=\frac{2m}{h^2}\int_0^h(h-y)y^2dy$$

$$=\frac{2m}{h^2}\int_0^h\left(hy^2-y^3\right)dy$$

$$=\frac{2m}{h^2}\left[\frac{hy^3}{3}-\frac{y^4}{4}\right]_0^h=\frac{2m}{h^2}\left[\frac{h^4}{3}-\frac{h^4}{4}\right]=\frac{2m}{h^2}\left[\frac{h^4}{12}\right]=\frac{mh^2}{6}$$

23. We have:

$$y=D\left[1-\cos\left(\frac{\pi x}{2L}\right)\right]$$

$$\frac{dy}{dx}=D\left(\frac{\pi}{2L}\right)\sin\left(\frac{\pi x}{2L}\right)$$

$$\frac{d^2y}{dx^2}=D\left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right)$$

(8.2) $\int dr/r=\ln|r|$

(8.26) $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$V = \frac{EI}{2}\int_0^L \left[D\left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right)\right]^2 dx$$

$$= \frac{EI\left(D\right)^2}{2}\left(\frac{\pi}{2L}\right)^4 \int_0^L \cos^2\left(\frac{\pi x}{2L}\right)dx \qquad (\dagger)$$

$$\int_0^L \cos^2\left(\frac{\pi x}{2L}\right)dx = \int_0^L \underbrace{\frac{1}{2}\left[1+\cos\left(\frac{2\pi x}{2L}\right)\right]}_{\text{by (4.68)}}dx = \frac{1}{2}\int_0^L \left[1+\cos\left(\frac{\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[x - \frac{\sin\left(\pi x/L\right)}{\pi/L}\right]_0^L = \frac{1}{2}\left[L - \frac{L\sin\left(\pi L/L\right)}{\pi} - 0\right] = \frac{L}{2}$$

Substituting this into $(\dagger)$

$$V = \frac{EID^2}{2}\left(\frac{\pi}{2L}\right)^4 \frac{L}{2} = \frac{EID^2\pi^4 L}{2^6 L^4} = \frac{EID^2\pi^4}{64L^3}$$

24.

$$p = \frac{\eta I^2 L^2 \pi}{4\lambda^2}\int_0^\pi \sin^3\left(\theta\right)d\theta \qquad (*)$$

Need to evaluate

$$\int_0^\pi \sin^3\left(\theta\right)d\theta = \int_0^\pi \sin\left(\theta\right)\underbrace{\left(1-\cos^2\left(\theta\right)\right)}_{\text{by (4.64)}}d\theta$$

Let $u = \cos\left(\theta\right) \qquad \frac{du}{d\theta} = -\sin\left(\theta\right) \qquad d\theta = \frac{du}{-\sin\left(\theta\right)}$

The limits are $u = \cos(\pi) = -1$ and $u = \cos(0) = 1$

Substituting these values

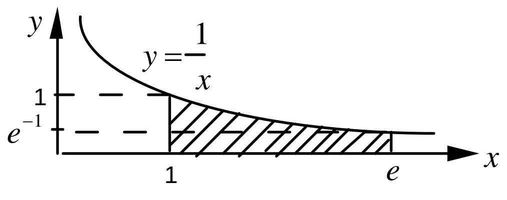
$$\int_0^\pi \sin(\theta)\left(1-\cos^2(\theta)\right)d\theta = \int_1^{-1} \sin(\theta)\left(1-u^2\right)\frac{du}{-\sin(\theta)}$$

$$= \int_1^{-1}\left(u^2-1\right)du = \left[\frac{u^3}{3} - u\right]_1^{-1} = \frac{4}{3}$$

Substituting this into (*) gives

$$p = \frac{\eta I^2 L^2 \pi}{4\lambda^2}.\frac{4}{3} = \frac{\eta I^2 L^2 \pi}{3\lambda^2}$$

25.



$$\int_1^e \frac{dx}{x} = \left[\ln\left(x\right)\right]_1^e = \ln\left(e\right) - \ln\left(1\right) = 1 - 0 = 1$$

(4.64) $\sin^2(\theta) + \cos^2(\theta) = 1$

(4.68) $\cos^2(A) = \frac{1}{2}\left[1 + \cos(2A)\right]$

26. MAPLE commands are shown on the web site.

27. Use (8.32) with $a=2$ and $u=x$:

$$\int_0^2 \left(4-x^2\right)^{1/2} dx = \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$=\left[\frac{2}{2}\sqrt{4-2^2}+\frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right)\right]-[0]=2\sin^{-1}(1)=2(\pi/2)=\pi$$

28. We have $u=a\cosh(\theta)$. Differentiating

$$\frac{du}{d\theta}\underbrace{=}_{(6.26)} a\sinh(\theta) \text{ gives } du=a\sinh(\theta)d\theta$$

Substituting $u=a\cosh(\theta)$ and $du=a\sinh(\theta)d\theta$ into the integral gives:

$$\int\frac{du}{\sqrt{u^2-a^2}}=\int\frac{a\sinh(\theta)d\theta}{\sqrt{a^2\cosh^2(\theta)-a^2}} \qquad (*)$$

The denominator, $\sqrt{a^2\cosh^2(\theta)-a^2}$, simplifies:

$$\sqrt{a^2\cosh^2(\theta)-a^2} = \sqrt{a^2\underbrace{\left(\cosh^2(\theta)-1\right)}_{=\sinh^2(\theta)}}=\sqrt{a^2\sinh^2(\theta)}=a\sinh(\theta)$$

Substituting this into (*) gives

$$\int\frac{du}{\sqrt{u^2-a^2}}=\int\frac{a\sinh(\theta)}{a\sinh(\theta)}d\theta=\int d\theta=\theta+C \qquad (**)$$

What is θ?

We know $a\cosh(\theta)=u$ so $\cosh(\theta)=\frac{u}{a}$ which gives $\theta=\cosh^{-1}\left(\frac{u}{a}\right)$.

Replacing θ with $\cosh^{-1}\left(\frac{u}{a}\right)$ into (**) displays our result:

$$\int\frac{du}{\sqrt{u^2-a^2}}=\cosh^{-1}\left(\frac{u}{a}\right)+C$$

29. We have $u=a\tanh(\theta)$

$$\frac{du}{d\theta}\underbrace{=}_{(6.27)} a\operatorname{sec}h^2(\theta) \text{ gives } du=a\operatorname{sec}h^2(\theta)d\theta$$

Substituting $u=a\tanh(\theta)$ and $du=\operatorname{sec}h^2(\theta)d\theta$ we have

$$\int\frac{du}{a^2-u^2}=\int\frac{a\operatorname{sec}h^2(\theta)d\theta}{a^2-a^2\tanh^2(\theta)} \qquad (*)$$

The denominator simplifies to:

$$a^2-a^2\tanh^2(\theta)=a^2\underbrace{\left(1-\tanh^2(\theta)\right)}_{\operatorname{sec}h^2(\theta)}=a^2\operatorname{sec}h^2(\theta)$$

(6.26) $\quad \left[\cosh(\theta)\right]'=\sinh(\theta)$

(8.32) $\quad \int\sqrt{a^2-u^2}du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{2}\right)$

Substituting this into (*) gives

$$\int \frac{du}{a^2-u^2} = \int \frac{a\sec h^2(\theta)\,d\theta}{a^2\sec h^2(\theta)} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$

We can obtain θ from the substitution

$$a\tanh(\theta) = u$$

$$\tanh(\theta) = \frac{u}{a} \text{ which gives } \theta = \tanh^{-1}\left(\frac{u}{a}\right)$$

Thus $\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C$

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30. How do we evaluate $\int_0^{2\pi} \omega t\cos(n\omega t)d(\omega t)$?

Use integration by parts

$$u = \omega t \qquad v' = \cos(n\omega t)$$

$$\frac{du}{d(\omega t)} = 1 \qquad v = \int \cos(n\omega t)\,d(\omega t) = \frac{\sin(n\omega t)}{n}$$

Using (8.45) gives

$$\int_0^{2\pi} \omega t\cos(n\omega t)\,d(\omega t) = \left[\omega t\frac{\sin(n\omega t)}{n}\right]_{\omega t=0}^{\omega t=2\pi} - \int_0^{2\pi}\frac{\sin(n\omega t)}{n}d(\omega t)$$

$$= \underbrace{\left[\frac{2\pi\sin(2\pi n)-0}{n}\right]}_{=0 \text{ because } \sin(2\pi n)=0} - \int_0^{2\pi}\frac{\sin(n\omega t)}{n}d(\omega t)$$

$$= 0 - \int_0^{2\pi}\frac{\sin(n\omega t)}{n}d(\omega t)$$

$$= -\left[-\frac{\cos(n\omega t)}{n^2}\right]_{\omega t=0}^{\omega t=2\pi} = \frac{1}{n^2}\left[\underbrace{\cos(2\pi n)}_{=1} - \underbrace{\cos(0)}_{=1}\right] = \frac{1}{n^2}[1-1] = 0$$

(6.27) $\left[\tanh(\theta)\right]' = \sec h^2(\theta)$

(8.45) $\int uv dt = u\int v dt - \int\left[du\int v dt\right]dt$