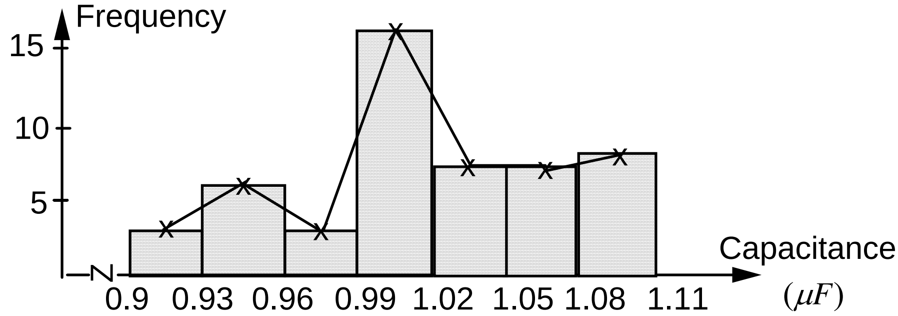
Complete solutions to Miscellaneous Exercise 16

1. We need to group the data. The smallest value is 0.901 and largest is 1.098, so we need to divide 0.9-1.1 into classes. Take class width =0.03, you could take 0.02 or 0.04 or some other suitable value. We choose 0.03 because it gives 7 groups. With too many classes in a histogram it is difficult to visualise the data.

Capacitance (μF)	Frequency
0.900-0.930	3
0.930-0.960	6
0.960-0.990	3
0.990-1.020	16
1.020-1.050	7
1.050-1.080	7
1.080-1.110	8



Histogram and frequency polygon showing the capacitance measurements of $1\mu F$.

2. Substituting the mean $\mu = 45.5$ and s.d. $\sigma = 5$ into (16.48) gives

$$z = \frac{x - 45.5}{5} \qquad (*)$$

(a) Putting $x = 35$ into (*) yields

$$z = \frac{35 - 45.5}{5} = -2.1$$

Using the normal distribution table with $z = 2.1$ gives 0.9821. Hence the probability of a student obtaining less than 35 marks is

$$1 - 0.9821 = 0.0179$$

(b) Substituting $x = 50$ into (*) gives

$$z = \frac{50 - 45.5}{5} = 0.9$$

Using the normal distribution table with $z = 0.9$ gives 0.8159. The probability of a student obtaining more than 50 marks is

$$1 - 0.8159 = 0.1841$$

(c) Substituting $x = 60$ into (*) gives

$$z = \frac{60 - 45.5}{5} = 2.9$$

(16.48) $$z = \frac{x - \mu}{\sigma}$$

The normal distribution table gives 0.99813 for $z = 2.9$. The probability of a student obtaining marks between 50 and 60 is $0.99813 - 0.8159 = 0.18223$.

3. We have

$$P(\text{at least one engine working to specification})$$
$$= 1 - P(\text{no engine working to specification})$$
$$= 1 - 0.1^2 = 0.99$$

4. (a) A stage is working if at least one of the circuits is activated, hence

$$P(\text{at least one of the circuits is activated})$$
$$= 1 - P(\text{no circuit is activated})$$
$$= 1 - 0.15^3 = 0.997 \text{ (3 d.p.)}$$

(b) $P(\text{both stages}) = 0.997 \times 0.997 = 0.993$ (3 d.p.)

5. (i) A student's birthday can be any of the 365 days and since we have 30 students so the total number of possibilities are 365^{30}.
The total number of permutations of choosing 30 days from 365 is ${}^{365}P_{30}$.

$$P(\text{no 2 students have their birthday on the same day})$$
$$= \frac{{}^{365}P_{30}}{365^{30}} = 0.294 \quad \text{(3 d.p.)}$$

MAPLE gives this answer without any trouble. Normally a calculator will fail because it cannot handle the large numbers.
(ii) Similarly

$$P(\text{no 2 students out of 60 have their birthday on the same day})$$
$$= \frac{{}^{365}P_{60}}{365^{60}} = 0.006 \quad \text{(3 d.p.)}$$

Yes.
There is less than 1% (0.6%) chance that no 2 students out of 60 will have their birthday on the same day. Your intuition tells you that the probability should be higher since there are 365 days.

6. Let X be the random variable representing the number of correct answers. We use the binomial distribution, (16.24), with

$p = \frac{1}{5} = 0.2,\ q = 0.8$ and $n = 10$:

(a) For all answers correct, $X = 10$

$$P(X = 10) = 0.2^{10} = 1.024 \times 10^{-7} \text{ (4 s.f.)}$$

(b) For none correct, $X = 0$

$$P(X = 0) = 0.8^{10} = 0.1074 \quad \text{(4 s.f.)}$$

(c) For exactly 4 correct, $X = 4$

$$P(X = 4) = {}^{10}C_4 (0.8)^6 (0.2)^4 = 0.0881 \text{ (4 d.p.)}$$

(16.24) $$P(X = x) = {}^{n}C_x p^x q^{n-x}$$

7. (i) Similar to question 6 of **EXERCISE 16(f)**. Substituting the given x values into $P(X=x)=\frac{x^2}{k}$ gives

$$P(X=0)=0$$
$$P(X=1)=\frac{1}{k}$$
$$P(X=2)=\frac{2^2}{k}=\frac{4}{k}$$
$$P(X=3)=\frac{3^2}{k}=\frac{9}{k}$$
$$P(X=4)=\frac{4^2}{k}=\frac{16}{k}$$

Total these upto 1

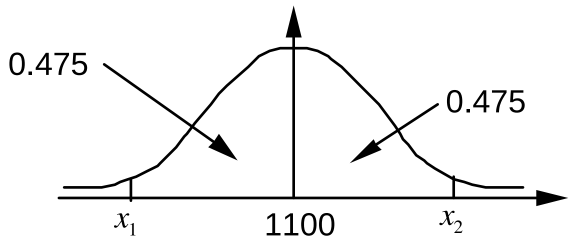
$$\frac{1}{k}+\frac{4}{k}+\frac{9}{k}+\frac{16}{k}=1$$
$$\frac{30}{k}=1 \text{ gives } k=30$$

Our probability distribution is

x	0	1	2	3	4
$P(X=x)$	0	1/30	4/30	9/30	16/30

Using a calculator we find
(ii) mean = 3.33 (2 d.p.) (iii) s.d. = 0.83 (2 d.p.) (iv) variance = 0.69 (2 d.p.)

8. Since we are looking at an area of 0.95 (95%) and so by symmetry the area in each half is $\frac{0.95}{2}=0.475$.



The area less than the x_2 value is $0.5+0.475=0.975$.
The z value for 0.975 from the tables is 1.96. Using $z=\frac{x-\mu}{\sigma}$ with $z=1.96$, $\mu=1100$, $\sigma=50$ gives

$$\frac{x_2-1100}{50}=1.96$$
$$x_2=1198 \text{ hours}$$

Similarly

$$\frac{x_1-1100}{50}=-1.96$$
$$x_1=1002 \text{ hours}$$

1002 to 1198 hours contains 95% of the wearout times.

9. Since $F(t)=1-R(t)$ we have

$$F(t)=1-\frac{k^2}{(k+t)^2}$$

Using $f(t)=\frac{dF}{dt}$ we have

$$f(t)=\frac{dF}{dt}=\frac{d}{dt}\left[1-\frac{k^2}{(k+t)^2}\right]$$

$$=\frac{d}{dt}\left[1-k^2(k+t)^{-2}\right]$$

$$=\frac{2k^2}{(k+t)^3}$$

We know $h(t)=\frac{f(t)}{R(t)}$, hence for $t\geq 0$

$$h(t)=\frac{2k^2/(k+t)^3}{k^2/(k+t)^2}=\frac{2}{k+t} \text{ (cancelling)}$$

---

10. From (16.45) we have

$$h(t)=\frac{f(t)}{R(t)} \qquad (\dagger)$$

Also

$$f(t)=\frac{dF(t)}{dt} \text{ and } F(t)=1-R(t)$$

$$\frac{dF(t)}{dt}=-\frac{dR(t)}{dt}$$

Hence $f(t)=-\frac{dR(t)}{dt}$. Let $R=R(t)$.

Substituting $h(t)=\lambda$ and $f(t)=-\frac{dR}{dt}$ into (†) gives

$$-\frac{dR}{Rdt}=\lambda$$

$$\frac{dR}{R}=-\lambda dt$$

Integrating both sides

$$\int\frac{dR}{R}=-\int\lambda dt$$

$$\ln[R]=-\lambda t$$

Taking exponential of both sides gives

$$R(t)=e^{-\lambda t}$$

---

11. We have

$$H(t) = -\int_0^t \frac{d}{dx}\left[\ln\left[R(x)\right]\right]dx$$

$$= -\left[\ln\left[R(x)\right]\right]_0^t$$

$$= -\left[\ln\left[R(t)\right]\right] + \ln\underbrace{\left[R(0)\right]}_{=1}$$

$$= -\ln\left[R(t)\right] \qquad (\text{remember } \ln(1) = 0)$$

---

12. Since 2% of transformers are defective, so we have

$$P(\text{no defective}) = 0.98$$

$$P(\text{no defective in a sample of size } n) = 0.98^n$$

We need to find n which gives

$$0.98^n = 0.8$$

Taking natural logs, $\ln$, gives

$$\ln\left(0.98^n\right) = \ln(0.8)$$

$$n\ln(0.98) = \ln(0.8)$$

$$n = \frac{\ln(0.8)}{\ln(0.98)} = 11.045$$

Hence $n = 11$, cannot be 12 because $0.98^{12} < 0.8$.

---

13. At $t = 0$, $f(t) = \dfrac{1}{2.5} = 0.4$ because the area under the density function $= 1$. Using the equation of a straight line with gradient $= -0.4/5$ and intercept $= 0.4$ we have

$$f(t) = 0.4 - \frac{0.4}{5}t = 0.4 - 0.08t \qquad 0 \le t \le 5$$

Using $F(t) = \int_0^t f(x)\,dx$ gives

$$F(t) = \int_0^t (0.4 - 0.08x)\,dx$$

$$= \left[0.4x - \frac{0.08x^2}{2}\right]_0^t = 0.4t - 0.04t^2$$

Using the definitions from reliability engineering we have

$$F(t) = 0.4t - 0.04t^2 \qquad 0 \le t \le 5$$

$$R(t) = 1 - 0.4t + 0.04t^2 \qquad 0 \le t \le 5$$

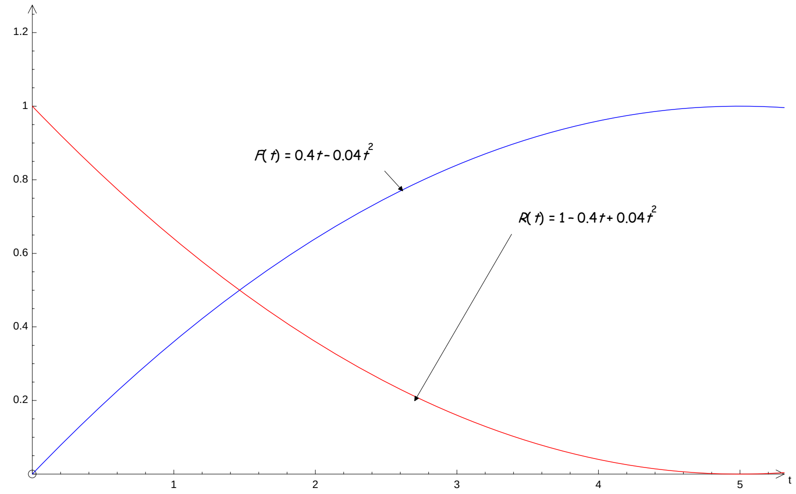
Using $MTTF = \int_0^\infty R(t)\,dt$ with $R(t) = 1 - 0.4t + 0.04t^2$ gives

$$MTTF = \int_0^5 \left(1 - 0.4t + 0.04t^2\right)dt$$

$$= \left[t - \frac{0.4t^2}{2} + \frac{0.04t^3}{3}\right]_0^5$$

$$= 5 - \left(\frac{0.4\times 5^2}{2}\right) + \left(\frac{0.04\times 5^3}{3}\right) = \frac{5}{3}\text{ years}$$

The graphs $F(t) = 0.4t - 0.04t^2$ and $R(t) = 1 - 0.4t + 0.04t^2$ are:



Note that at $t = 5$, $F(t) = 1$ and $R(t) = 0$.
$R(t)$ represents the probability of a component lasting upto t years. For $t > 5$, $R(t) = 0$ because the component does not last more than 5 years. $F(t)$ represents the probability of the component failing within t years. For $t > 5$, $F(t) = 1$, component will definitely fail within 5 years. We could rewrite these functions as

$$R(t) = \begin{cases} 1 - 0.4t + 0.04t^2 & 0 \le t \le 5 \\ 0 & t > 5 \end{cases} \quad \text{and} \quad F(t) = \begin{cases} 0.4t - 0.04t^2 & 0 \le t \le 5 \\ 1 & t > 5 \end{cases}$$

14.

```
> R[1]:=t->exp(-t);
```

$$R_1 := t \to e^{(-t)}$$

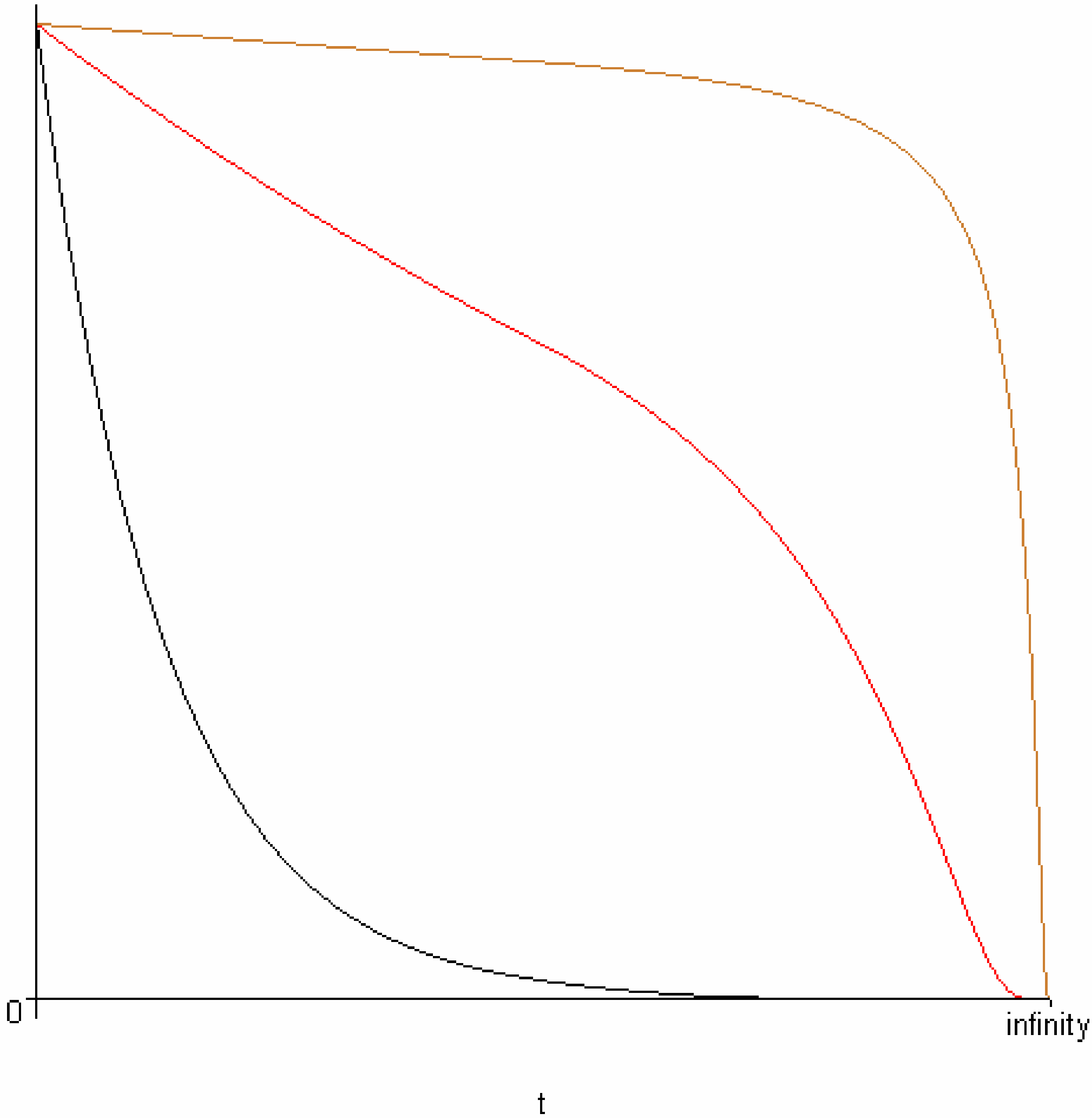
```
> R[2]:=t->exp(-0.1*t);
```

$$R_2 := t \to e^{(-0.1\,t)}$$

```
> R[3]:=t->exp(-0.01*t);
```

$$R_3 := t \to e^{(-0.01\,t)}$$

```
>
plot({R[1](t),R[2](t),R[3](t)},t=0..infinity,color=[black
,red,gold]);
```



The smaller the λ, the longer the component lasts. Yes because $R(0)=1$ means that the component is initially working. $\lim_{t\to\infty} R(t)=0$ means that the component will eventually fail.

15. The failure distribution function is determined by $F(t)=\int_0^t f(x)dx$ with $f(x)=\dfrac{4}{(1+x)^5}$.

$$F(t) = \int_0^t \left[\frac{4}{(1+x)^5} \right] dx$$

$$= 4\int_0^t (1+x)^{-5} dx$$

$$= 4\left[\frac{(1+x)^{-4}}{-4} \right]_0^t$$

$$= \frac{-1}{(1+t)^4} + 1$$

$$F(t) = 1 - \frac{1}{(1+t)^4}$$

$$R(t) = 1 - F(t) = 1 - \left(1 - \frac{1}{(1+t)^4} \right) = \frac{1}{(1+t)^4}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{4/(1+t)^5}{1/(1+t)^4} = \frac{4}{1+t}$$

(Since t is in years the above functions are defined for $t \geq 0$).

$$P(\text{a component lasts more than 10 years}) = R(10) = \frac{1}{(1+10)^4} = \frac{1}{14641}$$

Using MAPLE to plot $R(t)$ and $F(t)$ on the same axes:

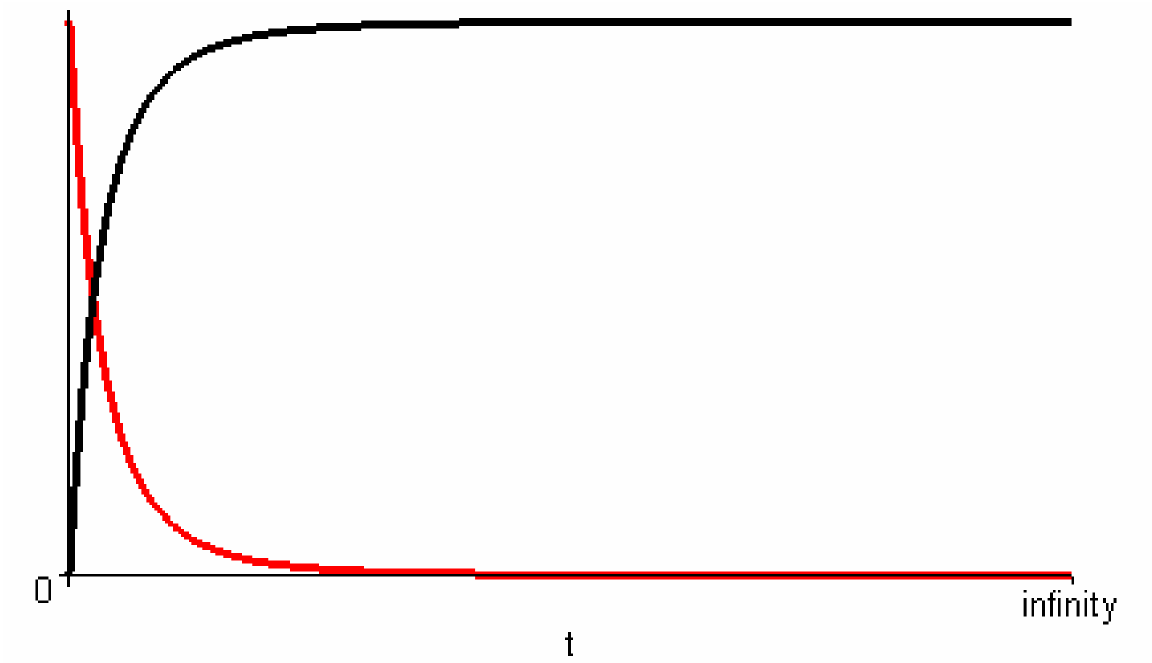
```
> R:=t->1/(1+t)^4;
```

$$R := t \rightarrow \frac{1}{(t+1)^4}$$

```
> F:=t->1-R(t);
```

$$F := t \rightarrow 1 - \mathrm{R}(t)$$

```
> plot({R(t),F(t)},t=0..infinity,color=[black,red]);
```



16. (a) We have

```
> F:=t->4-20/(5+t/3);
```

$$F := t \to 4 - \frac{20}{5 + \frac{1}{3}t}$$

```
> R:=simplify(1-F(t));
```

$$R := -\frac{3\,(-5+t)}{15+t}$$

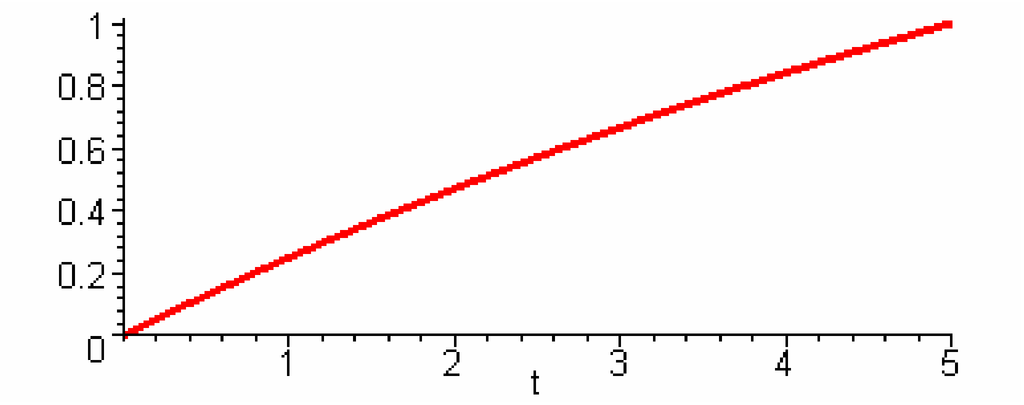
```
> f:=diff(F(t),t);
```

$$f := \frac{20}{3\left(5 + \frac{t}{3}\right)^2}$$

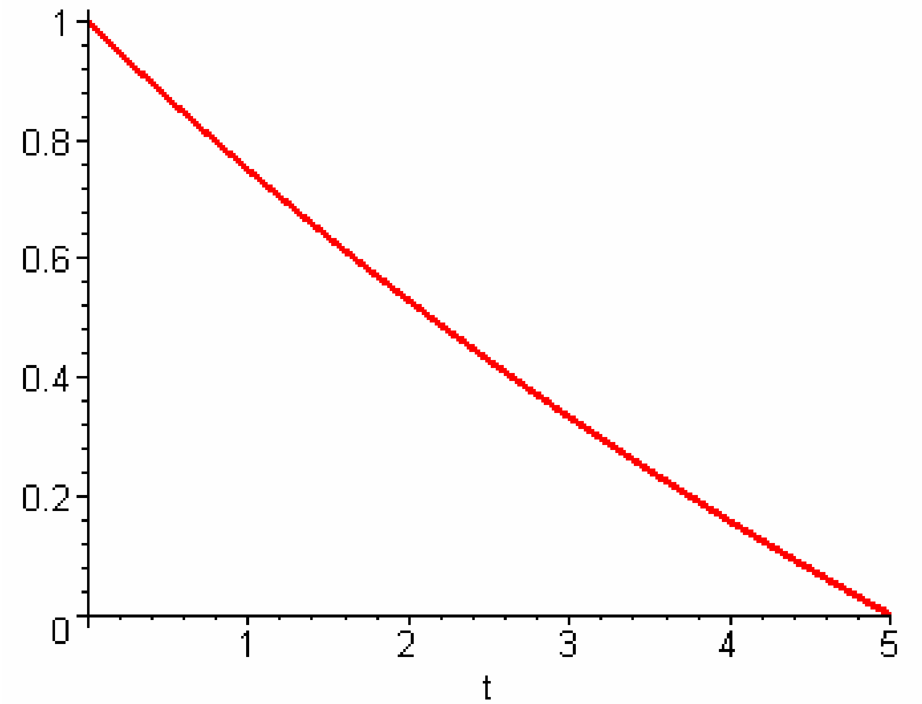
```
> h:=simplify(f/R);
```

$$h := -\frac{20}{(-5+t)\,(15+t)}$$

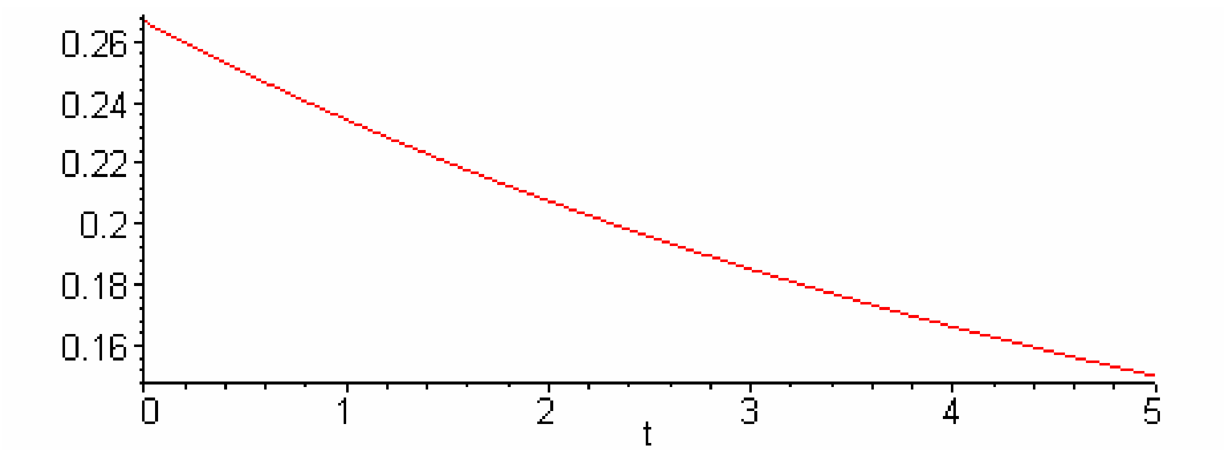
```
> plot(F(t),t=0..5);
```



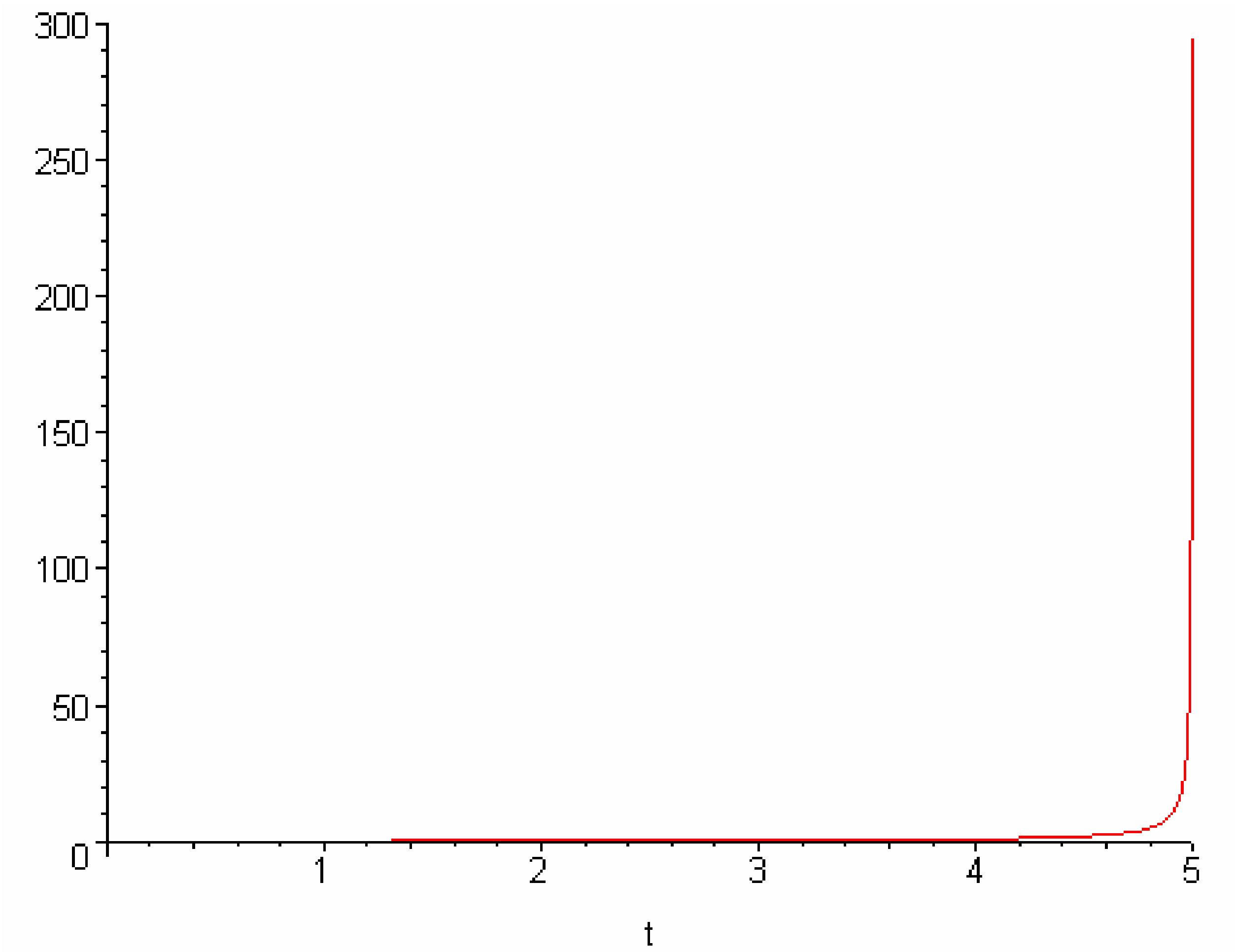

```
> plot(R,t=0..5);
```




```
> plot(f,t=0..5);
```




```
> plot(h,t=0..5);
```



(c) All the probabilities can be evaluated by using the reliability function, $R(t) = -3\left(\frac{-5+t}{15+t}\right)$. Substitute the corresponding t value into $R(t)$.

(i) $P(\text{component lasts more than 1 year}) = R(1) = -3\left(\frac{-5+1}{15+1}\right) = \frac{3}{4}$

(ii) $P(\text{component lasts more than 2 years}) = R(2) = -3\left(\frac{-5+2}{15+2}\right) = \frac{9}{17}$

(iii) $P(\text{component lasts more than 3 years}) = R(3) = -3\left(\frac{-5+3}{15+3}\right) = \frac{1}{3}$

(iv) $P(\text{component lasts more than 4 years}) = R(4) = -3\left(\frac{-5+4}{15+4}\right) = \frac{3}{19}$

(v) $P(\text{component lasts more than 5 years}) = R(5) = -3\left(\frac{-5+5}{15+5}\right) = 0$

(d) The zero means that a component cannot last more than 5 years.