Solutions to Miscellaneous Exercise 9

1. Let W = work done then

$$W = \int_{0.01}^{0.09} PdV$$

Using Simpson's rule (9.4) we have

$$W \approx \frac{0.01}{3}\{946 + 4[358 + 136 + 77 + 51] + 2[203 + 99 + 62] + 44\} = 14.02 \text{ kJ}$$

2. Using Simpson's rule (9.4) again:

$$W \approx \frac{0.025}{3}\{10.06 + 4[4.23 + 1.78 + 1.07 + 0.75] + 2[2.55 + 1.34 + 0.88] + 0.64\}$$
$$= 0.43 \text{ kJ} = 430 \text{ J}$$

3. In this case we cannot use Simpson's rule because we need an even number of intervals so we apply the trapezium rule.

$$\Delta h \approx \frac{50}{2}\left[3.57 + 2(3.63 + 3.7 + 3.75 + 3.8) + 3.86\right] = 930 \text{ kJ / kg} \quad (3 \text{ s.f.})$$

4. Establishing a table

t	0.0	2.5	5.0	7.5	10.0
$5\ln\left\lvert\frac{2}{2+3t}\right\rvert + 9.81t$	0	16.73	38.35	61.05	84.24

Using Simpson's rule:

$$h \approx \frac{2.5}{3}\left[0 + 4(16.73 + 61.05) + 2(38.35) + 84.24\right] = 393 \text{ m} \quad (3 \text{ s.f.})$$

5. Let's apply the Simpson's rule with width $h = 10$.

v	50	60	70	80	90
$1/(5 - 2v^{0.11})$	0.52	0.54	0.55	0.57	0.58

$$t \approx \frac{10}{3}\left[0.52 + 4(0.54 + 0.57) + 2(0.55) + 0.58\right] = 22.1 \text{ s} \quad (3 \text{ s.f.})$$

6. Apply Simpson's rule with $h = 0.1$:

t	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\sin(t)/t$	0.97	0.96	0.94	0.92	0.90	0.87	0.84

By (9.7) the mean value, M, is given by

$$M = \frac{1}{1-0.4}\int_{0.4}^{1} \frac{\sin(t)}{t}\,dt$$
$$\underbrace{\approx}_{\text{using Simpson}} \frac{1}{0.6}\left[\frac{0.1}{3}\{0.97 + 4(0.96 + 0.92 + 0.87) + 2(0.94 + 0.9) + 0.84\}\right] = 0.92 \text{ V}$$

(9.4) $$\int_a^b y\,dx \approx \frac{h}{3}\{y_0 + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots) + y_n\}$$

(9.7) Mean value of $y = \frac{1}{b-a}\int_a^b y\,dx$

7. We use numerical integration to find the mean value of i over the interval π to $3\pi/2$. Use Simpson's rule with $h=\pi/8$:

t	π	$9\pi/8$	$10\pi/8$	$11\pi/8$	$12\pi/8$
$i=\sqrt{1-2\sin(t)}$	1	1.33	1.55	1.69	1.73

Applying (9.7) gives

$$\text{Mean value of } i=\frac{1}{3\pi/2-\pi}\int_{\pi}^{3\pi/2}\sqrt{1-2\sin(t)}\,dt$$
$$\approx\frac{2}{\pi}\left\{\frac{\pi/8}{3}\left[1+4(1.33+1.69)+2(1.55)+1.73\right]\right\}$$
$$=\frac{1}{12}[17.91]=1.49\text{A}$$

8. We have

$$\text{Area }=\int_{-3}^{3}\sqrt{1-\frac{x^2}{9}}dx$$
$$=\int_{-3}^{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\,dx$$
$$=3\left[\frac{x}{6}\sqrt{1-\left(\frac{x}{3}\right)^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}\qquad\left(\text{By (8.32), } u=\frac{x}{3}\text{ and } a=1\right)$$
$$=3\left[\frac{3}{6}\sqrt{1-1^2}+\frac{1}{2}\sin^{-1}\left(\frac{3}{3}\right)\right]-3\left[-\frac{3}{6}\sqrt{1-(-1)^2}+\frac{1}{2}\sin^{-1}\left(-\frac{3}{3}\right)\right]$$
$$=\frac{3}{2}\left[\sin^{-1}(1)-\sin^{-1}(-1)\right]\qquad(\text{Simplifying})$$
$$=\frac{3}{2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=\frac{3}{2}[\pi]=\frac{3\pi}{2}$$

(Since the function is even we could integrate between 0 and 3 and then double our result).

9. Using (9.7)

$$\text{Mean value of } v=\frac{1}{20}\int_0^{20}5\left(1-e^{-t/4}\right)dt$$
$$=\frac{1}{4}\int_0^{20}\left(1-e^{-t/4}\right)dt\qquad\left(\text{Because }\frac{5}{20}=\frac{1}{4}\right)$$
$$=\frac{1}{4}\left[t+\frac{e^{-t/4}}{1/4}\right]_0^{20}\qquad(\text{Integrating})$$
$$=\frac{1}{4}\left[\left(20+4e^{-5}\right)-\left(0+4e^{0}\right)\right]=4.01\text{V}\ (2\text{ d.p.})$$

(8.32) $$\int\sqrt{a^2-u^2}\,du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)$$

(9.7) $$\text{Mean value of } y=\frac{1}{b-a}\int_a^b y\,dx$$

Let $v_{R.M.S}$ represent the $R.M.S$. value of v:

$$\left(v_{R.M.S}\right)^2 = \frac{1}{20}\int_0^{20} 5^2\left(1-e^{-t/4}\right)^2 dt$$

$$= \frac{5}{4}\int_0^{20}\left(1-2e^{-t/4}+e^{-2t/4}\right)dt \qquad \left(\text{Because } \frac{25}{20}=\frac{5}{4}\right)$$

$$= \frac{5}{4}\left[t+\frac{2e^{-t/4}}{1/4}-\frac{e^{-t/2}}{1/2}\right]_0^{20} \qquad (\text{Integrating})$$

$$= \frac{5}{4}\left[\left(20+8e^{-5}-2e^{-10}\right)-\left(0+8e^{0}-2e^{0}\right)\right]=17.567 \text{ V}$$

To find the $R.M.S$. value we take the square root:

$$v_{R.M.S.} = \sqrt{17.567} = 4.19 \text{ V} \quad (3 \text{ s.f.})$$

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10. (a) Integrating gives

$$v = \int\left[\frac{10}{10-t}+9.8\right]dt$$

$$= \int\frac{10dt}{10-t}+\int 9.8dt$$

$$v = -10\ln(10-t)+9.8t+C$$

Substituting $t=0,\ v=0$

$$0 = -10\ln(10)+C \text{ gives } C = 10\ln(10)$$

Thus we have

$$v = 10\ln(10)-10\ln(10-t)+9.8t$$

$$=10\left[\ln(10)-\ln(10-t)\right]+9.8t \qquad (\text{Taking Out } 10)$$

$$v = 10\ln\left(\frac{10}{10-t}\right)+9.8t$$

(b) We use numerical integration for integrating v to find the height.
Using Simpson's rule with interval =1

t	0	1	2	3	4	5	6
$v = 10\ln\left(\frac{10}{10-t}\right)+9.8t$	0	10.85	21.83	32.97	44.31	55.93	67.96

$$\text{height} \approx \frac{1}{3}\left[0+4(10.85+32.97+55.93)+2(21.83+44.31)+67.96\right]$$

$$= 199.75 = 200 \text{ m} \quad (3 \text{ s.f.})$$

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11. We have $P = 1961V^{-1.41}$

$$\text{Work done } = \int_{0.013}^{0.09} 1961V^{-1.41}dV = 1961\int_{0.013}^{0.09} V^{-1.41}dV$$

$$= 1961\left[\frac{V^{-0.41}}{-0.41}\right]_{0.013}^{0.09}$$

$$= \frac{1961}{-0.41}\left[0.09^{-0.41}-0.013^{-0.41}\right]$$

$$= 15540.96J = 15.5 \text{ kJ} \quad (3 \text{ s.f.})$$

12. Using (9.13) we have

$$\frac{d^2y}{dx^2}=\frac{1}{EI}\left[2.5(x-5)^2+8(x-3)-5x\right]$$
$$=\frac{1}{EI}\left[2.5(x^2-10x+25)+8x-24-5x\right]$$
$$=\frac{1}{EI}\left[2.5x^2-25x+62.5+3x-24\right]$$
$$\frac{d^2y}{dx^2}=\frac{1}{EI}\left[2.5x^2-22x+38.5\right]$$

Integrating twice:

$$\frac{dy}{dx}=\frac{1}{EI}\left[\frac{2.5x^3}{3}-\frac{22x^2}{2}+38.5x\right]+C$$
$$y=\frac{1}{EI}\left[\frac{2.5x^4}{12}-\frac{11x^3}{3}+\frac{38.5x^2}{2}\right]+Cx+D \qquad (*)$$

Substituting $x=0,\ y=0$ into (*)

$$0=\frac{1}{EI}[0]+C(0)+D \quad \text{gives } D=0$$

Substituting $x=5,\ y=0$ into (*)

$$0=\frac{1}{EI}\left[\frac{2.5\times5^4}{12}-\frac{11\times5^3}{3}+\frac{38.5\times5^2}{2}\right]+5C$$
$$=\frac{1}{EI}[153.125]+5C$$
$$C=-\frac{153.125}{5EI}=-\frac{30.625}{EI}$$

Putting $C=-\dfrac{30.625}{EI}$ and $D=0$ into (*)

$$y=\frac{1}{EI}\left[\frac{2.5x^4}{12}-\frac{11x^3}{3}+\frac{38.5x^2}{2}-30.625x\right]$$

14. (i) We have

$$\frac{d^2x}{dt^2}=0$$

Integrating gives

$$\frac{dx}{dt}=C \text{ where } C \text{ is a constant.}$$

Using $t=0,\ \dfrac{dx}{dt}=v\cos(\theta)$ gives $C=v\cos(\theta)$

$$\frac{dx}{dt}=v\cos(\theta)$$

Integrating again with respect to t gives

$$x=v\cos(\theta)t+D \text{ where } D \text{ is a constant}$$

(9.13) $$\frac{d^2y}{dx^2}=\frac{M}{EI}$$

Using $t=0,\ x=0$ gives $D=0$. Hence

$$x=\left[v\cos\left(\theta\right)\right]t$$

(ii) We have

$$\frac{d^2y}{dt^2}=-g$$

Integrating gives

$$\frac{dy}{dt}=-gt+k \qquad \left(k=\text{integration constant}\right)$$

Using $t=0,\ \dfrac{dy}{dt}=v\sin(\theta)$ gives $k=v\sin(\theta)$

We have

$$\frac{dy}{dt}=-gt+v\sin(\theta)$$

Integrating again gives

$$y=-\frac{gt^2}{2}+\left[v\sin\left(\theta\right)\right]t+E$$

Using $t=0,\ y=0$ gives $E=0$. Hence

$$y=\left[v\sin\left(\theta\right)\right]t-\frac{gt^2}{2}$$

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15. Substituting $a=0,\ b=\pi/\omega$ into (9.8) gives

$$\left(v_{R.M.S.}\right)^2=\frac{\omega}{\pi}\int_0^{\pi/\omega}v^2dt \qquad (\dagger)$$

To show the result in the question, we need only evaluate $\int_0^{\pi/\omega}v^2dt$ and substitute our answer into (†).

$$\begin{aligned}v^2&=\left[V_1\sin\left(\omega t\right)+V_3\sin\left(3\omega t\right)\right]^2\\&=\left[V_1\sin\left(\omega t\right)\right]^2+2V_1V_3\sin\left(\omega t\right)\sin\left(3\omega t\right)+\left[V_3\sin\left(3\omega t\right)\right]^2\\v^2&=V_1^2\sin^2\left(\omega t\right)+V_3^2\sin^2\left(3\omega t\right)+2V_1V_3\sin\left(\omega t\right)\sin\left(3\omega t\right)\end{aligned}$$

Substituting this into (†) and separating the integrals gives

$$\left(v_{R.M.S.}\right)^2=\frac{\omega}{\pi}\left[\begin{aligned}V_1^2\int_0^{\pi/\omega}\sin^2\left(\omega t\right)dt+V_3^2\int_0^{\pi/\omega}\sin^2\left(3\omega t\right)dt\\+V_1V_3\int_0^{\pi/\omega}2\sin\left(\omega t\right)\sin\left(3\omega t\right)dt\end{aligned}\right] \qquad (\dagger\dagger)$$

How do we evaluate the first integral in (††)*?*

We can use (4.68):

$$\int_0^{\pi/\omega}\sin^2\left(\omega t\right)dt=\int_0^{\pi/\omega}\frac{1}{2}\left[1-\cos\left(2\omega t\right)\right]dt$$

(9.8) $$\left(v_{R.M.S.}\right)^2=\frac{1}{b-a}\int_a^b v^2dt$$

(4.68) $$\sin^2\left(A\right)=\frac{1}{2}\left[1-\cos\left(2A\right)\right]$$

$$= \frac{1}{2}\int_0^{\pi/\omega}\left[1-\cos\left(2\omega t\right)\right]dt$$

$$= \frac{1}{2}\left[t-\frac{\sin\left(2\omega t\right)}{2\omega}\right]_0^{\pi/\omega}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{\omega}-\underbrace{\frac{\sin\left(2\omega\left(\frac{\pi}{\omega}\right)\right)}{2\omega}}_{=0}\right]-0\right\} = \frac{\pi}{2\omega}$$

Similarly $\int_0^{\pi/\omega}\sin^2\left(3\omega t\right)dt = \dfrac{\pi}{2\omega}$.

How do we integrate

$$2\sin\left(3\omega t\right)\sin\left(\omega t\right)$$

with respect to t*?*

Use (4.59) and then integrate the result.

$$\int_0^{\pi/\omega} 2\sin\left(3\omega t\right)\sin\left(\omega t\right)dt = \int_0^{\pi/\omega}\left[\cos\left(3\omega t-\omega t\right)-\cos\left(3\omega t+\omega t\right)\right]dt$$

$$= \int_0^{\pi/\omega}\left[\cos\left(2\omega t\right)-\cos\left(4\omega t\right)\right]dt$$

$$= \left[\frac{\sin\left(2\omega t\right)}{2\omega}-\frac{\sin\left(4\omega t\right)}{4\omega}\right]_0^{\pi/\omega}$$

$$= \left[\underbrace{\frac{\sin\left(2\omega\left(\frac{\pi}{\omega}\right)\right)}{2\omega}}_{=0}-\underbrace{\frac{\sin\left(4\omega\left(\frac{\pi}{\omega}\right)\right)}{4\omega}}_{=0}\right]-0=0$$

Putting $\int_0^{\pi/\omega}\sin^2\left(\omega t\right)dt = \dfrac{\pi}{2\omega}$, $\int_0^{\pi/\omega}\sin^2\left(3\omega t\right)dt = \dfrac{\pi}{2\omega}$ and

$\int_0^{\pi/\omega} 2\sin(3\omega t)\sin(\omega t)dt = 0$ into $(\dagger\dagger)$ gives

$$\left(v_{R.M.S.}\right)^2 = \frac{\omega}{\pi}\left[V_1^2\left(\frac{\pi}{2\omega}\right)+V_3^2\left(\frac{\pi}{2\omega}\right)+V_1V_2\left(0\right)\right]$$

$$= \frac{\omega}{\pi}\left(\frac{\pi}{2\omega}\right)\left[V_1^2+V_3^2\right] = \frac{1}{2}\left[V_1^2+V_3^2\right]$$

Thus the result required.

(4.59) $$2\sin\left(A\right)\sin\left(B\right) = \cos\left(A-B\right)-\cos\left(A+B\right)$$