Complete Solutions to Exercise 11(g)

1. (a) (i) We are given the matrix $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ for which we need to find the eigenvalues and corresponding eigenvectors.

$$\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\det\begin{pmatrix}1-\lambda & 0\\ 0 & 2-\lambda\end{pmatrix}$$

$$=\left(1-\lambda\right)\left(2-\lambda\right)=0 \quad \text{gives} \quad \lambda_1=1 \text{ and } \lambda_2=2$$

For the eigenvalue $\lambda_1=1$:

$$\begin{pmatrix}1-1 & 0\\ 0 & 2-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{gives } x=1 \text{ and } y=0$$

Our eigenvector for $\lambda_1=1$ is $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$. The eigenvector for the other eigenvalue $\lambda_2=2$ is

$$\begin{pmatrix}1-2 & 0\\ 0 & 2-2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{gives } x=0 \text{ and } y=1$$

The eigenvector corresponding to $\lambda_2=2$ is $\mathbf{v}=\begin{pmatrix}0\\ 1\end{pmatrix}$.

(ii) To find the invertible (has an inverse) matrix **P** we need to follow the procedure outlined in the main text.

<u>Step 1:</u>

The eigenvector are $\mathbf{u}=\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}0\\ 1\end{pmatrix}$.

<u>Step 2:</u>

The matrix **P** is given by

$$\mathbf{P}=\left(\mathbf{u} \vdots \mathbf{v}\right)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$$

Note that the matrix **P** is the identity matrix **I**.

<u>Step 3 and 4:</u>

The diagonal matrix **D** is given by $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$:

$$\mathbf{D}=\mathbf{I}^{-1}\mathbf{AI}=\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

Note that **D** is a diagonal matrix with **eigenvalues** $\lambda_1=1$ and $\lambda_2=2$ as the entries along the leading diagonal.

Note that for 2 by 2 matrices it is quicker to show $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$ rather than $\mathbf{PD}=\mathbf{AP}$.

(b) (i) We need to find the eigenvalues and corresponding eigenvectors of $\mathbf{A}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\begin{pmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix}$$
$$=(1-\lambda)(1-\lambda)-1$$
$$=1-2\lambda+\lambda^2-1$$
$$=\lambda^2-2\lambda=\lambda(\lambda-2)=0 \text{ gives } \lambda_1=0 \text{ and } \lambda_2=2$$

For the eigenvalue $\lambda_1=0$ we can find the eigenvector by:

$$\begin{pmatrix}1-0 & 1\\ 1 & 1-0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x+y=0$$

Thus $x=-y$ and let $y=1$ then $x=-1$. The corresponding eigenvector is $\mathbf{u}=\begin{pmatrix}-1\\ 1\end{pmatrix}$.

Similarly we can find the eigenvector **v** belonging to the other eigenvalue $\lambda_2=2$:

$$\begin{pmatrix}1-2 & 1\\ 1 & 1-2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ gives } x=1 \text{ and } y=1$$

Thus $\mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$ belongs to the eigenvalue $\lambda_2=2$.

(ii) Step 1:

The eigenvectors are $\mathbf{u}=\begin{pmatrix}-1\\ 1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$.

Step 2:

Let $\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v})=\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$.

Step 3 and 4:

Then taking the inverse of this matrix **P** we get $\mathbf{P}^{-1}=\frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$. Thus $\mathbf{D}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is

$$\mathbf{D}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\frac{1}{2}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}$$

Again the leading diagonal entries are 0 and 2 which are the eigenvalues of **A**.

(c) (i) The eigenvalues of $\mathbf{A}=\begin{pmatrix}3 & 0\\ 4 & 4\end{pmatrix}$ are given by

$$\det(\mathbf{A}-\lambda\mathbf{I})=\det\begin{pmatrix}3-\lambda & 0\\ 4 & 4-\lambda\end{pmatrix}$$
$$=(3-\lambda)(4-\lambda) \text{ gives } \lambda_1=3 \text{ and } \lambda_2=4$$

The eigenvector **u** belonging to $\lambda_1=3$ is

$$\begin{pmatrix} 3-3 & 0 \\ 4 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 1 \text{ and } y = -4$$

Hence $\mathbf{u} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. The eigenvector $\mathbf{v}$ corresponding to $\lambda_2 = 4$ is given by

$$\begin{pmatrix} 3-4 & 0 \\ 4 & 4-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 0 \text{ and } y = 1$$

Thus $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $\lambda_2 = 4$.

(ii) Step 1:

The eigenvectors are $\mathbf{u} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step 2:

Let $\mathbf{P} = (\mathbf{u} \vdots \mathbf{v}) = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$.

Step 3 and 4:

Then taking the inverse of this matrix **P** we get $\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. Thus $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is

$$\begin{aligned} \mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

Again the leading diagonal entries are 3 and 4 which are the eigenvalues of **A**.

(d) (i) We are given the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. The eigenvalues are

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (2-\lambda)(3-\lambda) - 2 \\ &= \lambda^2 - 5\lambda + 4 = 0 \text{ gives } \lambda_1 = 1 \text{ and } \lambda_2 = 4 \end{aligned}$$

The eigenvector **u** belonging to $\lambda_1 = 1$ is

$$\begin{pmatrix} 2-1 & 2 \\ 1 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 2 \text{ and } y = -1$$

Hence $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. The eigenvector $\mathbf{v}$ corresponding to $\lambda_2 = 4$ is given by

$$\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 1 \text{ and } y = 1$$

Thus $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\lambda_2 = 4$.

(ii) Step 1:

The eigenvectors are $\mathbf{u}=\begin{pmatrix}2\\-1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}1\\1\end{pmatrix}$.

Step 2:

Let $\mathbf{P}=\left(\mathbf{u}\ \vdots\ \mathbf{v}\right)=\begin{pmatrix}2 & 1\\-1 & 1\end{pmatrix}$.

Step 3 and 4:

Then taking the inverse of this matrix **P** we get $\mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}$. Thus $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$ is

$$\begin{aligned}\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}&=\frac{1}{3}\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}2 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 1\end{pmatrix}\\&=\frac{1}{3}\begin{pmatrix}1 & -1\\4 & 8\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 1\end{pmatrix}\\&=\frac{1}{3}\begin{pmatrix}3 & 0\\0 & 12\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\end{aligned}$$

Again the leading diagonal entries are 1 and 4 which are the eigenvalues of **A**.

2. We have $\mathbf{A}^m=\mathbf{PD}^m\mathbf{P}^{-1}$ and we use this to find $\mathbf{A}^5$.

(a) We are given $\mathbf{A}=\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$ and from question 1 part (a) we have $\mathbf{P}=\mathbf{P}^{-1}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\mathbf{I}$ and $\mathbf{D}=\mathbf{A}=\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$. Therefore

$$\mathbf{A}^5=\mathbf{PD}^5\mathbf{P}^{-1}=\mathbf{I}\begin{pmatrix}1^5 & 0\\0 & 2^5\end{pmatrix}\mathbf{I}=\begin{pmatrix}1 & 0\\0 & 32\end{pmatrix}$$

Of course this confirms our earlier work on matrices that the power of a diagonal matrix is the matrix with the diagonal entries to the power and zeros everywhere else.

$$\text{If }\mathbf{A}=\begin{pmatrix}a_1 & 0 & 0\\0 & \ddots & 0\\0 & 0 & a_n\end{pmatrix}\text{ then }\mathbf{A}^m=\begin{pmatrix}a_1^{\ m} & 0 & 0\\0 & \ddots & 0\\0 & 0 & a_n^{\ m}\end{pmatrix}$$

(b) We are given $\mathbf{A}=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$ and from question 1 part (b) we have $\mathbf{P}=\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix}$, $\mathbf{P}^{-1}=\frac{1}{2}\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}$. Thus

$$\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$$
$$= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}^5 \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 32 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 0 & 32 \\ 0 & 32 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 32 & 32 \\ 32 & 32 \end{pmatrix} = \begin{pmatrix} 16 & 16 \\ 16 & 16 \end{pmatrix}$$

(c) We need to find $\mathbf{A}^5$ for $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 4 & 4 \end{pmatrix}$. From question 1 part (c) we have

$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$, $\mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. Thus

$$\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$$
$$= \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}^5 \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 243 & 0 \\ 0 & 1024 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 243 & 0 \\ -972 & 1024 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 243 & 0 \\ 3124 & 1024 \end{pmatrix}$$

(d) We need to find $\mathbf{A}^5$ given that $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. We use $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ with $m = 5$. *What is* ***P**, **D** and* $\mathbf{P}^{-1}$ *equal to?*

By question 1 part (d) we have $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, $\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. Thus substituting these into $\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$ gives

$$\mathbf{A}^5 = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^5 \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 2 & 1024 \\ -1 & 1024 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 1026 & 2046 \\ 1023 & 2049 \end{pmatrix} = \begin{pmatrix} 342 & 682 \\ 341 & 683 \end{pmatrix}$$

3. (a) (i) *What are the eigenvalues of the diagonal matrix* $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$*?*

1, 2 and 3. *What are the corresponding eigenvectors?*

For $\lambda_1 = 1$ let **u** be the eigenvector then

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix}1-1 & 0 & 0\\ 0 & 2-1 & 0\\ 0 & 0 & 3-1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \text{gives } x=1,\ y=0 \text{ and } z=0$$

Thus $\mathbf{u}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$ is the eigenvector belonging to $\lambda_1=1$. Let $\mathbf{v}$ be the eigenvector belonging to the second eigenvalue $\lambda_2=2$:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix}1-2 & 0 & 0\\ 0 & 2-2 & 0\\ 0 & 0 & 3-2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \text{gives } x=0,\ y=1 \text{ and } z=0$$

Hence the eigenvector $\mathbf{v}=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$ corresponds to the eigenvalue $\lambda_2=2$. Let **w** be the eigenvector belonging to $\lambda_3=3$:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{u}=\begin{pmatrix}1-3 & 0 & 0\\ 0 & 2-3 & 0\\ 0 & 0 & 3-3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \text{gives } x=0,\ y=0 \text{ and } z=1$$

The eigenvector $\mathbf{w}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$ belongs to the eigenvalue $\lambda_3=3$.

Step 1:

The eigenvectors are $\mathbf{u}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$.

Step 2:

The invertible (has an inverse) matrix is $\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v}\ \vdots\ \mathbf{w})=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I}$.

Step 3 and 4:

Hence $\mathbf{P}^{-1} = \mathbf{I}^{-1} = \mathbf{I}$. We have

$$\mathbf{D} = \mathbf{I}^{-1}\mathbf{A}\mathbf{I} = \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Again the eigenvalues $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$ of the matrix **A** are along the leading diagonal.

(iii) To find $\mathbf{A}^4$ we need to use $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ with $m = 4$:

$$\mathbf{A}^4 = \mathbf{I}\mathbf{D}^4\mathbf{I}^{-1} = \mathbf{D}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

(b) (i) *What are the eigenvalues of* $\mathbf{A} = \begin{pmatrix} -1 & 4 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{pmatrix}$ *?*

The eigenvalues are $a_1 = -1$, $a_2 = 4$ and $a_3 = 5$. *What else do we need to find?*

The eigenvector for each eigenvalue. Let **u** be the eigenvector belonging to $a_1 = -1$:

$$(\mathbf{A} - a_1\mathbf{I})\mathbf{u} = \begin{pmatrix} -1-(-1) & 4 & 0 \\ 0 & 4-(-1) & 3 \\ 0 & 0 & 5-(-1) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{gives} \quad x = 1,\ y = 0 \text{ and } z = 0$$

Thus $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the eigenvector belonging to $a_1 = -1$. Let **v** be the eigenvector belonging to the second eigenvalue $a_2 = 4$. We have

$$(\mathbf{A} - a_2\mathbf{I})\mathbf{v} = \begin{pmatrix} -1-4 & 4 & 0 \\ 0 & 4-4 & 3 \\ 0 & 0 & 5-4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{gives} \quad x = 4,\ y = 5 \text{ and } z = 0$$

Thus $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$. Let **w** be the eigenvector belonging to $a_3 = 5$. We have

$$(\mathbf{A}-\lambda_3\mathbf{I})\mathbf{w}=\begin{pmatrix}-1-5 & 4 & 0\\ 0 & 4-5 & 3\\ 0 & 0 & 5-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=\begin{pmatrix}-6 & 4 & 0\\ 0 & -1 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{gives} \quad x=2,\ y=3 \text{ and } z=1$$

Thus $\mathbf{w}=\begin{pmatrix}2\\3\\1\end{pmatrix}$. We have the eigenvectors $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}4\\5\\0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}2\\3\\1\end{pmatrix}$ corresponding to the eigenvalues $\lambda_1=-1,\ \lambda_2=4$ and $\lambda_3=5$ respectively.

(ii) Step 1:

The eigenvectors are $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}4\\5\\0\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}2\\3\\1\end{pmatrix}$.

Step 2:

The invertible (has an inverse) matrix is $\mathbf{P}=(\mathbf{u}\ \vdots\ \mathbf{v}\ \vdots\ \mathbf{w})=\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}$.

Step 3 and 4:

The diagonal matrix $\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}$.

We could check that we have the correct matrix **P** by checking $\mathbf{PD}=\mathbf{AP}$ but we need to determine $\mathbf{A}^4$ for part (iii) so we have to find the inverse of **P**.

Using MATLAB or our early theory on matrices we have $\mathbf{P}^{-1}=\dfrac{1}{5}\begin{pmatrix}5 & -4 & 2\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}$.

Evaluating

$$\mathbf{D}=\mathbf{P}^{-1}\mathbf{AP}=\frac{1}{5}\begin{pmatrix}5 & -4 & 2\\ 0 & 1 & -3\\ 0 & 0 & 5\end{pmatrix}\begin{pmatrix}-1 & 4 & 0\\ 0 & 4 & 3\\ 0 & 0 & 5\end{pmatrix}\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}-5 & 4 & -2\\ 0 & 4 & -12\\ 0 & 0 & 25\end{pmatrix}\begin{pmatrix}1 & 4 & 2\\ 0 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-5 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 25\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5\end{pmatrix}$$

(iii) To find $\mathbf{A}^4$ we need to use $\mathbf{A}^m=\mathbf{PD}^m\mathbf{P}^{-1}$ with $m=4$:

$$\mathbf{A}^4 = \mathbf{PD}^4\mathbf{P}^{-1} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}^4 \frac{1}{5} \begin{pmatrix} 5 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 625 \end{pmatrix} \begin{pmatrix} 5 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 1024 & 1250 \\ 0 & 1280 & 1875 \\ 0 & 0 & 625 \end{pmatrix} \begin{pmatrix} 5 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 5 & 1020 & 3180 \\ 0 & 1280 & 5535 \\ 0 & 0 & 3125 \end{pmatrix} = \begin{pmatrix} 1 & 204 & 636 \\ 0 & 256 & 1107 \\ 0 & 0 & 625 \end{pmatrix}$$