Complete solutions to Exercise 3(d)

1. (i) $f(2x+3)=(2x+3)+1=2x+4$

(ii) $f(x)\cdot g(x)=(x+1)(2x+3)=\underset{F}{2x^2}+\underset{O}{3x}+\underset{I}{2x}+\underset{L}{3}$

$$=2x^2+5x+3$$

(iii) $g-1=(2x+3)-1=2x+2$

(iv) $f+g=(x+1)+(2x+3)=3x+4$

(v) $f-g=(x+1)-(2x+3)=x+1-2x-3=-x-2$

(vi) $\dfrac{f}{g}=\dfrac{x+1}{2x+3}\quad(2x+3\neq 0)$

2. (i) $f\circ f=f(f(x))=f(2x+3)=2(2x+3)+3$

$$=4x+9$$

(ii) $f\left(f(f(x))\right)=f(\underset{\uparrow}{4x+9})=2(4x+9)+3=8x+21$

because $f(f(x))=4x+9$ by part (i)

(iii) $f(f(f(-3)))\underbrace{=}_{\substack{\text{by result (ii)}\\ \text{with } x=-3}}[8\times(-3)]+21=-3$

3. (i)

$$f(0)=(a\times 0^2)+(b\times 0)+c=c$$

(ii) Replace x with $x+1$ into $f(x)$:

$$f(x+1)=a(x+1)^2+b(x+1)+c$$

$$=\underbrace{a(x^2+2x+1)}_{\text{by (1.13)}}+bx+b+c$$

$$=ax^2+2ax+a+bx+b+c$$

$$=ax^2+(2a+b)x+a+b+c$$

(iii) $f(x+1)-f(x)=\underbrace{ax^2+2ax+bx+a+b+c}_{=f(x+1)}-(ax^2+bx+c)$

$$=ax^2+2ax+bx+a+b+c-ax^2-bx-c$$

$$=2ax+a+b$$

4. (i)

$$g(g(x))=g(x+1)=(x+1)+1=x+2$$

(ii)

$$f(g(x))=f(x+1)=(x+1)^2\underbrace{=}_{\text{By (1.13)}}x^2+2x+1$$

(iii) $g(f(x))=g(x^2)=x^2+1$

5. (i) $f(g(x))=f(x)=x^2-1$

(ii) $g(f(x))=g(x^2-1)=x^2-1=f(x)$

(iii) Since g is the identity function $f\circ g=g\circ f=f$.

Note: It is always the case if f or g are identity functions then $f\circ g=g\circ f$.

(iv) $g\circ g=g(x)=x$

(1.13) $\qquad (a+b)^2=a^2+2ab+b^2$

(v) We have

$$f\circ f = f(x^2-1) = (x^2-1)^2 - 1$$
$$= \underbrace{x^4 - 2x^2 + 1}_{\text{by (1.14)}} - 1$$
$$= x^4 - 2x^2$$
$$= x^2(x^2-2)$$

6. (i) Let $y = f(x)$ and then transpose to make x the subject:

$$y = \frac{6}{3-x}$$
$$3y - xy = 6$$
$$3y - 6 = xy$$
$$x = \frac{3y-6}{y}$$
$$f^{-1}(x) = \frac{3x-6}{x} \quad (x \neq 0)$$

(ii)

$$f\circ f^{-1} = f\left(\frac{3x-6}{x}\right) = \frac{6}{3-\left(\dfrac{3x-6}{x}\right)} \underbrace{=}_{\text{multiply numerator and denominator by } x} \frac{6x}{3x-3x+6} = \frac{\cancel{6}x}{\cancel{6}} = x$$

(iii)

$$f^{-1}\circ f = f^{-1}\left(\frac{6}{3-x}\right) = \frac{3\cdot\left(\dfrac{6}{3-x}\right)-6}{6/(3-x)}$$
$$= \frac{18-6(3-x)}{6} \quad (\text{multiply numerator and denominator by } 3-x)$$
$$= \frac{18-18+6x}{6} = \frac{\cancel{6}x}{\cancel{6}} = x$$

7. (i) $f\left(f^{-1}(x)\right) = f\left(\left(\dfrac{2x-3}{1+x}\right)^{\frac{1}{3}}\right) = \dfrac{\left(\dfrac{2x-3}{1+x}\right)^{\frac{1}{3}\times 3} + 3}{2-\left(\dfrac{2x-3}{1+x}\right)^{\frac{1}{3}\times 3}}$

$$= \frac{\dfrac{2x-3}{1+x}+3}{2-\dfrac{2x-3}{1+x}} = \frac{2x-3+3(1+x)}{2(1+x)-(2x-3)} \quad \text{(multiplying top and bottom by } (1+x))$$
$$= \frac{2x-3+3+3x}{2+2x-2x+3}$$
$$= \frac{5x}{5} = x$$

(1.14) $\quad (a-b)^2 = a^2 - 2ab + b^2$

(ii)

$$f^{-1} \circ f = f^{-1}\left(\frac{x^3+3}{2-x^3}\right)$$

$$= \left(\frac{2\left(\frac{x^3+3}{2-x^3}\right)-3}{1+\frac{x^3+3}{2-x^3}}\right)^{\frac{1}{3}}$$

$$= \left(\frac{2x^3+6-3(2-x^3)}{2-x^3+x^3+3}\right)^{\frac{1}{3}}$$

$$= \left(\frac{2x^3+6-6+3x^3}{5}\right)^{\frac{1}{3}}$$

$$= \left(\frac{\cancel{5}x^3}{\cancel{5}}\right)^{\frac{1}{3}} = x$$

8. We have $f \circ f^{-1}(x) = f^{1} \circ f(x) = x$ (identity function). This is generally the case.

9. (i) $\sqrt{x}-1 = f\left(\sqrt{x}\right) = f\left(g(x)\right) = f \circ g$ or $\sqrt{x}-1 = g(x)-1$. Hence $f \circ g = g-1$

(ii) We have

$$\sqrt{x-1} = g(x-1) = g\big(f(x)\big) = g \circ f$$

(iii) We have

$$\sqrt{x-1}+7 = \underbrace{g \circ}_{\text{by (ii)}} f + 7$$

(iv) $\sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = x-1 = f$ or $\sqrt{(x-1)^2} = g\big((x-1)^2\big) = g\left(f^2\right)$

10. Very similar to **EXAMPLE 16**; $F(t) = \frac{t}{5}$, $R(t) = 1-\frac{t}{5}$ and $h(\mathrm{t}) = \frac{1}{5-t}$.

11. (a) Substituting $G(s) = \frac{k}{s(s+1)}$ and $\mathrm{N(s)} = 0.01$ into

$$T(s) = \frac{G(s)}{1+N(s)G(s)}$$

gives

$$T(s) = \frac{\frac{k}{s(s+1)}}{1+0.01\left(\frac{k}{s(s+1)}\right)} \qquad (\dagger)$$

Simplify (†) by multiplying numerator and denominator by $s(s+1)$:

$$T(s) = \frac{k}{s(s+1)+0.01k}$$

$$= \frac{k}{s^2+s+0.01k}$$

(b) Similarly for (b) we multiply numerator and denominator by $s+k_1$:

$$T(s)=\frac{\frac{1}{(s+k_1)}}{1+k_2\left(\frac{1}{s+k_1}\right)}$$

$$=\frac{1}{(s+k_1)+k_2}$$

$$T(s)=\frac{1}{s+(k_1+k_2)}$$

(c) Note that we can factorize the denominator of G(s):

$$G(s)=\frac{s+1}{s^2+3s+2}$$

$$=\frac{s+1}{(s+2)(s+1)}$$

$$G(s)=\frac{1}{s+2}\quad \text{(cancelling } s+1)$$

We have $G(s)=\frac{1}{s+2}$ and $N(s)=0.3$, this is of the form of (b) where $G(s)=\frac{1}{s+k_1}$ and $N(s)=k_2$. Here we have $k_1=2$ and $k_2=0.3$. Substituting $k_1=2$ and $k_2=0.3$ into the result for (b), $T(s)=\frac{1}{s+(k_1+k_2)}$, gives

$$T(s)=\frac{1}{s+(2+0.3)}=\frac{1}{s+2.3}$$

12.

$$G(s)=\frac{10s}{(s-2)(s^2+2s-5)}=\frac{10s}{s^3+2s^2-5s-2s^2-4s+10}$$

$$=\frac{10s}{s^3-9s+10}$$

$$\frac{G(s)}{1+G(s)H(s)}=\frac{10s/(s^3-9s+10)}{1+\left(\frac{10s}{s^3-9s+10}\right)\times(s+3)}$$

$$=\frac{10s}{(s^3-9s+10)+10s(s+3)}\quad \text{(multiplying top and bottom by } s^3-9s+10)$$

$$=\frac{10s}{s^3-9s+10+10s^2+30s}$$

$$=\frac{10s}{s^3+10s^2+21s+10}$$