Complete Solutions to Exercise 4(c)

1. To find θ we take inverse trigonometric function and then use CAST to find the other angle.

(a) $\theta = \sin^{-1}(0.707)$ which gives $\theta = 45°,\ 180° - 45° = 135°$

(b) $\theta = \cos^{-1}(0.5)$ which gives $\theta = 60°,\ 360° - 60° = 300°$

(c) $\theta = \tan^{-1}\left(\sqrt{3}\right)$ which gives $\theta = 60°,\ 180° + 60° = 240°$

(d) We have $\tan^2(\theta) = 3$, taking the square root of both sides gives $\tan\theta = \pm\sqrt{3}$. If $\tan(\theta) = +\sqrt{3}$ then $\theta = 60°,\ 240°$ (from (c)).

If $\tan(\theta) = -\sqrt{3}$ then by using a calculator $\theta = -60°$. However θ needs to be between $0°$ and $360°$, so $\theta = 360° - 60° = 300°$. The other angle can be found by applying CAST: $\theta = 180° - 60° = 120°$. Thus the angles satisfying the equation are:

$$\theta = 60°,\ 120°,\ 240°,\ 300°$$

(e) Let $x = \cos(\theta)$ then we have $x^2 - x + \frac{1}{4} = 0$. How do we solve this equation?

Factorizes into $\left(x - \frac{1}{2}\right)^2 = 0$ which means $x = \frac{1}{2}$ so $\cos(\theta) = \frac{1}{2}$. From (b) we have $\theta = 60°$ and $300°$.

(f) Note that $\sin^2(\theta) = \left[\sin(\theta)\right]^2$. Factorizing

$$2\sin^2(\theta) - 3\sin(\theta) = 2\left[\sin(\theta)\right]^2 - 3\sin(\theta) = 0$$

gives:

$$\left(2\sin(\theta) - 3\right)\sin(\theta) = 0$$

$$\sin(\theta) = \frac{3}{2} \quad \text{or} \quad \sin(\theta) = 0$$

No angle θ satisfies $\sin(\theta) = \frac{3}{2}$ because the sine function lies between -1 and +1.

So $\sin(\theta) = 0$ which gives $\theta = 0°, 180°$ and $360°$ (by graph of Fig 20).

(g) By transposing we have $\sin(\theta) = \frac{3}{7}$; taking inverse sin of both sides gives $\theta = 25.38°$ and the other angle is $180° - 25.38° = 154.62°$.

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2. (For all solutions to this question, n is any whole number and α is the angle found by using a calculator).

(a) The angle $\alpha = \tan^{-1}(0.8) = 38.66°$

By (4.18), the general solution is $\theta = (180 \times n)° + 38.66°$

(b) The angle $\alpha = 97.18°$. By (4.17) the general solution is

$$\theta = (360 \times n)° \pm 97.18°$$

(c) The angle $\alpha = 12.71°$. By (4.16) the general solution is

$$\theta = (180 \times n)° + \left[(-1)^n \times 12.71\right]°$$

(d) The angle $\alpha = \sin^{-1}(0.72) = 46.05°$. By (4.16)

$$3\theta = (180 \times n)° + \left[(-1)^n \times 46.05\right]°$$

$$\theta = (60 \times n)° + \left[(-1)^n \times 15.35\right]° \qquad \text{(dividing by 3)}$$

(4.16) $\theta = (180 \times n)° + (-1)^n \alpha$

(4.17) $\theta = (360 \times n)° \pm \alpha$

(4.18) $\theta = (180 \times n)° + \alpha$

(e) There is **no** solution to $\cos(2\theta + 60°) = 2$ because the cosine function lies between -1 and +1.

(f) $\cos(2\theta + 60°) = \frac{1}{2}$. We have $\alpha = \cos^{-1}\left(\frac{1}{2}\right) = 60°$, by (4.17),

$$2\theta + 60° = (360 \times n)° \pm 60°$$

$$2\theta = (360 \times n)° \pm 60° - 60° \quad (\text{Subtracting } 60°)$$

$$= (360 \times n)° \text{ or } (360 \times n)° - 120°$$

$$\theta = (180 \times n)° \text{ or } (180 \times n)° - 60° \quad (\text{Dividing by } 2)$$

3.(a) Let $x = \tan(\theta)$, then we have by rearranging

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0 \quad \text{gives} \quad x = 1$$

So $\tan(\theta) = 1$. $[\text{Replacing } x \text{ with } \tan(\theta)]$. With $\tan(\theta) = 1$, $\alpha = \tan^{-1}(1) = 45°$.

By (4.18) the general solution is $\theta = (180 \times n)° + 45°$. Substituting $n = 0,\ 1,\ 2$ and 3 gives:

$$\theta = 45°,\ \theta = 180° + 45°,\ \theta = (180 \times 2)° + 45° \text{ and } \theta = (180 \times 3)° + 45°$$

$$\theta = 45°,\ 225°,\ 405° \text{ and } 585°$$

(b) $\cos(\theta)\sin(\theta) + \cos(\theta) = 0$. Since $\cos(\theta)$ is common we factorize the equation:

$$(\sin(\theta) + 1)\cos(\theta) = 0$$

$$\cos(\theta) = 0 \quad \text{or} \quad \sin(\theta) = -1$$

For $\cos(\theta) = 0$, we have $\theta = 90°,\ 270°$ (by graph of Fig 21(a)).

For $\sin(\theta) = -1$, $\theta = \sin^{-1}(-1) = 270°$. Combining all the angles: $\theta = 90°,\ 270°$.

(c) Let $x = \cos(\theta)$, then we have

$$2x^2 - 4x - 5 = 0$$

How do we solve this quadratic equation?

By using (1.16) with $a = 2$, $b = -4$ and $c = -5$:

$$x = \frac{4 \pm \sqrt{16 + 40}}{4} = 1 \pm \frac{\sqrt{56}}{4} = -0.87 \text{ or } 2.87$$

Replacing x with $\cos(\theta)$ gives

$$\cos(\theta) = -0.87 \text{ or } \cos(\theta) = 2.87$$

$\cos(\theta) = 2.87$ is **not** a valid result because $\cos(\theta)$ lies between -1 and +1. Hence $\cos(\theta) = -0.87$ and taking inverse cos, $\theta = \cos^{-1}(-0.87) = 150.5°$.

(d) Substituting $x = \tan(\theta)$ gives:

$$x^2 - x - 1 = 0$$

We solve this quadratic equation by using (1.16) with $a = 1$, $b = -1$ and $c = -1$:

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = -0.62 \text{ or } 1.62$$

$$\tan(\theta) = -0.62 \text{ or } \tan(\theta) = 1.62$$

(1.16) $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$

(4.16) $\theta = (180 \times n)° + (-1)^n \alpha$

(4.17) $\theta = (360 \times n)° \pm \alpha$

(4.18) $\theta = (180 \times n)° + \alpha$

With $\tan(\theta) = -0.62$, $\alpha = \tan^{-1}(-0.62) = -31.80°$. By (4.18) the general solution is $\theta = (180 \times n)° + (-31.80°) = (180 \times n)° - 31.80°$. Substituting $n = 1$ and 2 gives:

$\theta = 180° - 31.80° = 148.2°$ and $\theta = (180 \times 2)° - 31.80° = 328.2°$ respectively.

With $\tan(\theta) = 1.62$, $\alpha = 58.31°$. By (4.18) the general solution is $\theta = (180 \times n)° + 58.31°$. Substituting $n = 0$ and 1 gives $\theta = 58.31°,\ 238.31°$. The result is $\theta = 58.31°,\ 148.20°,\ 238.31°,\ 328.20°$.

(e) Let $x = \sin(\theta)$ then solving the rearranged equation $x^2 - 6x + 4 = 0$ gives $x = 5.24,\ 0.76$. There is **no** value of θ such that $\sin(\theta) = 5.24$ so $\sin(\theta) = 0.76$

Hence applying (4.16) with

$$\alpha = \sin^{-1}(0.76) = 49.81°$$

gives the general solution:

$$\theta = (180 \times n)° + \left[(-1)^n \times 49.81\right]°$$

Substituting $n = 0,\ 1,\ 2$ and 3 respectively gives:

$$\theta = (180 \times 0)° + \left[(-1)^0 \times 49.81\right]°,\ (180 \times 1)° + \left[(-1)^1 \times 49.81\right]°,$$

$$(180 \times 2)° + \left[(-1)^2 \times 49.81\right]° \text{ and } (180 \times 3)° + \left[(-1)^3 \times 49.81\right]°$$

$\theta = 49.81°,\ 130.19°,\ 409.81°$ and $490.19°$.

(f) Let $x = \cos(\theta)$ then we have $x^2 - 10x + 23 = 0$ which gives

$$x = 3.58,\ x = 6.41$$

There is **no** solution to this equation because $x = \cos(\theta)$ can only lie between -1 and +1.

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4. (a) Remember $\cos^3(\theta) = [\cos(\theta)]^3$. Taking out a common factor of $\cos(\theta)$ gives $(2\cos^2(\theta) - 1)\cos(\theta) = 0$. Solving for $\cos(\theta)$:

$$\cos(\theta) = 0 \text{ or } \cos(\theta) = \pm\frac{1}{\sqrt{2}}$$

We have $\cos(\theta) = 0$ which gives $\theta = 90°,\ 270°$ and $\cos(\theta) = +\frac{1}{\sqrt{2}}$ gives q=$45°,\ 315°$. For $\cos(\theta) = -\frac{1}{\sqrt{2}}$ gives $\theta = 135°,\ 225°$.

Combining all the angles: $\theta = 45°, 90°, 135°, 225°, 270°$ and $315°$.

(b) Putting $x = \sin(\theta)$ into the rearranged equation

$$4\sin^2(\theta) - \sqrt{6}\sin(\theta) + \sqrt{2}\sin(\theta) = 0$$

gives:

$$4x^2 - \sqrt{6}x + \sqrt{2}x = 0$$
$$x\left(4x - \sqrt{6} + \sqrt{2}\right) = 0$$
$$x = 0 \text{ or } x = \frac{\sqrt{6} - \sqrt{2}}{4}$$

For $\sin(\theta) = 0$ gives $\theta = 0°,\ 180°,\ 360°$. For $\sin(\theta) = \frac{\sqrt{6} - \sqrt{2}}{4}$, we have by using a calculator:

(1.16) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(4.16) $\theta = (180 \times n)° + (-1)^n \alpha$

(4.18) $\theta = (180 \times n)° + \alpha$

$$\alpha = \sin^{-1}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = 15°$$

Applying CAST gives $\theta = 165°$. The angles satisfying the given equation are

$$\theta = 0°, 15°, 165°, 180°, 360°.$$

(c) Let $x = \cos(\theta)$, $y = \sin(\theta)$ Then we have

$$16x^2y^2 - 12x^2 - 12y^2 + 9 = 0$$

$$(4x^2 - 3)(4y^2 - 3) = 0 \text{ (Factorizing)}$$

$$x = \pm\frac{\sqrt{3}}{2} \qquad y = \pm\frac{\sqrt{3}}{2} \text{ (Solving)}$$

Replacing x with $\cos(\theta)$ and y with $\sin(\theta)$:

$$\cos(\theta) = +\frac{\sqrt{3}}{2} \quad \text{gives} \quad \theta = 30°,\ 330°$$

$$\cos(\theta) = -\frac{\sqrt{3}}{2} \quad \text{gives} \quad \theta = 150°,\ 210°$$

$$\sin(\theta) = +\frac{\sqrt{3}}{2} \quad \text{gives} \quad \theta = 60°,\ 120°$$

$$\sin(\theta) = -\frac{\sqrt{3}}{2} \quad \text{gives} \quad \theta = 240°,\ 300°$$

$$\theta = 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°$$