Complete solutions to Exercise 4(g)

1. (i) $\sin(75°)=\sin(45°+30°)\underbrace{=}_{\text{by (4.37)}}\left[\sin(45°)\cos(30°)\right]+\left[\cos(45°)\sin(30°)\right]$

$$\underbrace{=}_{\text{by TABLE 1}}\left(\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right)$$

$$=\frac{1}{2\sqrt{2}}\left(\sqrt{3}+1\right)\qquad\left(\text{factorizing }\frac{1}{2\sqrt{2}}\right)$$

(ii) For $\sin(120°)=\sin(90°+30°)$ apply (4.37) as above and we obtain $\frac{\sqrt{3}}{2}$.

(iii) $\cos(105°)=\cos(60°+45°)\underbrace{=}_{\text{by (4.39)}}\left[\cos(60°)\cos(45°)\right]-\left[\sin(60°)\sin(45°)\right]$

$$=\left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right)-\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\right)\qquad\text{(by TABLE 1)}$$

$$=\frac{1}{2\sqrt{2}}\left(1-\sqrt{3}\right)$$

(iv) Rewrite $\cos(150°)$ as $\cos(90°+60°)$ and then apply (4.39) which gives $-\frac{\sqrt{3}}{2}$.

(v) $\tan(15°)=\tan(45°-30°)\underbrace{=}_{\text{by (4.42)}}\dfrac{\tan(45°)-\tan(30°)}{1+\left[\tan(45°)\cdot\tan(30°)\right]}=\dfrac{1-\frac{1}{\sqrt{3}}}{1+\left(\frac{1}{\sqrt{3}}\right)}$.

Multiplying the numerator and denominator by $\sqrt{3}$ gives:

$$\tan(15°)=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

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2. (i) $\sin(90°-A)\underbrace{=}_{\text{by (4.38)}}\left(\underbrace{\sin(90°)}_{=1}\cos(A)\right)-\left(\underbrace{\cos(90°)}_{=0}\sin(A)\right)=\cos(A)-0=\cos(A)$

(ii) $\sin(-A)=\sin(0°-A)\underbrace{=}_{\text{by (4.38)}}\left(\underbrace{\sin(0°)}_{=0}\cos(A)\right)-\left(\underbrace{\cos(0°)}_{=1}\sin(A)\right)=0-\sin(A)=-\sin(A)$

(iii) $\cos(-A)=\cos(0°-A)\underbrace{=}_{\text{by (4.40)}}\left[\cos(0°)\cos(A)\right]+\left[\sin(0°)\sin(A)\right]=\cos(A)$

(iv) $\tan(-A)\underbrace{=}_{\text{by (4.35)}}\dfrac{\sin(-A)}{\cos(-A)}=\dfrac{-\sin(A)}{\cos(A)}=-\tan(A)$

(4.35)	$\tan(A)=\sin(A)/\cos(A)$
(4.37)	$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$
(4.38)	$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$
(4.39)	$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$
(4.40)	$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$
(4.42)	$\tan(A-B)=\left[\tan(A)-\tan(B)\right]/\left[1+\tan(A)\tan(B)\right]$

3. (i)

$$\cos(2\theta)=\cos(\theta+\theta)\underbrace{=}_{(4.39)}\cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)=[\cos(\theta)]^2-[\sin(\theta)]^2=\cos^2(\theta)-\sin^2(\theta)$$

(ii) $$\tan(2\theta)=\tan(\theta+\theta)\underbrace{=}_{\text{by }(4.41)}\frac{\tan(\theta)+\tan(\theta)}{1-[\tan(\theta)\cdot\tan(\theta)]}=\frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

4. We can replace $\cot(\theta)$ with $\sin(\theta)$ and $\cos(\theta)$: $\cot(\theta)=\dfrac{\cos(\theta)}{\sin(\theta)}$

So $\cot^2(\theta)=\dfrac{\cos^2(\theta)}{\sin^2(\theta)}$, we have

$$1+\cot^2(\theta)=1+\frac{\cos^2(\theta)}{\sin^2(\theta)}=\frac{\sin^2(\theta)+\cos^2(\theta)}{\sin^2(\theta)}\underbrace{=}_{\text{by }(4.64)}\frac{1}{\sin^2(\theta)}=\operatorname{cosec}^2(\theta)$$

$$\left(\text{because}\quad \operatorname{cosec}(\theta)=\frac{1}{\sin(\theta)}\right)$$

5. (i) $$\frac{1}{2}[1+\cos(2\theta)]=\frac{1}{2}\left(1+\underbrace{[\cos^2(\theta)-\sin^2(\theta)]}_{\text{by }(4.54)}\right)$$

$$=\frac{1}{2}\left(1+\cos^2(\theta)-\underbrace{[1-\cos^2(\theta)]}_{\text{by rearranging }(4.64)}\right)$$

$$=\frac{1}{2}[1+\cos^2(\theta)-1+\cos^2(\theta)]=\frac{1}{2}[2\cos^2(\theta)]=\cos^2(\theta)$$

(ii) Similar to part (i):

$$\frac{1}{2}[1-\cos(2\theta)]=\frac{1}{2}[1-(\cos^2(\theta)-\sin^2(\theta))]=\frac{1}{2}[1-[1-\sin^2(\theta)]+\sin^2(\theta)]=\sin^2(\theta)$$

6. (i) $$\sin(3\theta)=\sin(2\theta+\theta)$$

$$\underbrace{=}_{\text{by }(4.37)}\sin(2\theta)\cos(\theta)+\cos(2\theta)\sin(\theta)$$

$$=\underbrace{2\sin(\theta)\cos(\theta)}_{\text{by }(4.53)}\cos(\theta)+\underbrace{[1-2\sin^2(\theta)]}_{\text{by }(4.54)}\sin(\theta)$$

$$=2\sin(\theta)\cos^2(\theta)+(\sin(\theta)-2\sin^3(\theta))$$

$$=2\sin(\theta)\underbrace{[1-\sin^2(\theta)]}_{\text{by }(4.64)}+[\sin(\theta)-2\sin^3(\theta)]=2\sin(\theta)-2\sin^3(\theta)+\sin(\theta)-2\sin^3(\theta)$$

$$=3\sin(\theta)-4\sin^3(\theta)$$

(4.37)	$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$
(4.39)	$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$
(4.41)	$\tan(A+B)=[\tan(A)+\tan(B)]/[1-\tan(A)\tan(B)]$
(4.53)	$\sin(2A)=2\sin(A)\cos(A)$
(4.54)	$\cos(2A)=\cos^2(A)-\sin^2(A)=1-2\sin^2(A)$
(4.64)	$\cos^2(A)+\sin^2(A)=1$

(ii) Similar to (i).

(iii) $\tan(3\theta)=\tan(2\theta+\theta)$

$$\underbrace{=}_{\text{by }(4.41)}\frac{\tan(2\theta)+\tan(\theta)}{1-\tan(2\theta)\cdot\tan(\theta)}\underbrace{=}_{\text{by }(4.55)}\frac{\left(\frac{2t}{1-t^2}\right)+t}{1-\left(\frac{2t}{1-t^2}\right)\cdot t}\quad\text{where } t=\tan(\theta)$$

$$=\frac{2t+t(1-t^2)}{(1-t^2)-(2t\cdot t)}\quad\left(\begin{array}{l}\text{multiplying numerator and}\\ \text{denominator by } 1-t^2\end{array}\right)$$

$$=\frac{2t+t-t^3}{1-t^2-2t^2}=\frac{3t-t^3}{1-3t^2}\qquad (t=\tan(\theta))$$

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7. (i)

$$\sin(A+B)-\sin(A-B)=\underbrace{\sin A\cos B+\cos A\sin B}_{\text{by }(4.37)}-\underbrace{(\sin A\cos B-\cos A\sin B)}_{\text{by }(4.38)}=2\cos A\sin B$$

(ii)

$$\cos(A+B)+\cos(A-B)=\underbrace{\cos A\cos B-\sin A\sin B}_{\text{by }(4.39)}+\underbrace{\cos A\cos B+\sin A\sin B}_{\text{by }(4.40)}=2\cos A\cos B$$

(iii)

$$\cos(A-B)-\cos(A+B)=\underbrace{\cos A\cos B+\sin A\sin B}_{\text{by }(4.40)}-\underbrace{(\cos A\cos B-\sin A\sin B)}_{\text{by }(4.39)}=2\sin A\sin B$$

8. (i)

$$2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\underbrace{=}_{\text{by }(4.56)}\sin\left(\frac{A+B+A-B}{2}\right)+\sin\left(\frac{A+B-(A-B)}{2}\right)=\sin(A)+\sin(B)$$

(ii)

$$2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\underbrace{=}_{\text{by }(4.57)}\sin\left(\frac{A+B+A-B}{2}\right)-\sin\left(\frac{A+B-(A-B)}{2}\right)=\sin(A)-\sin(B)$$

(iii)

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\underbrace{=}_{\text{by }(4.58)}\cos\left(\frac{A+B+A-B}{2}\right)+\cos\left(\frac{A+B-(A-B)}{2}\right)=\cos(A)+\cos(B)$$

(iv)

$$2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)\underbrace{=}_{\text{by }(4.59)}\cos\left(\frac{A+B-(B-A)}{2}\right)-\cos\left(\frac{A+B+B-A}{2}\right)=\cos(A)-\cos(B)$$

(4.37)	$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$
(4.38)	$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$
(4.39)	$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$
(4.40)	$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$
(4.41)	$\tan(A+B)=[\tan(A)+\tan(B)]/[1-\tan(A)\tan(B)]$
(4.56)	$2\sin(A)\cos(B)=\sin(A+B)+\sin(A-B)$
(4.57)	$2\cos(A)\sin(B)=\sin(A+B)-\sin(A-B)$
(4.58)	$2\cos(A)\cos(B)=\cos(A+B)+\cos(A-B)$
(4.59)	$2\sin(A)\sin(B)=\cos(A-B)-\cos(A+B)$