Complete solutions to Exercise 4(i)

1. Use (4.75) on question 1.

(i) $R=\sqrt{10^2+10^2}=\sqrt{200}$ and $\beta=\tan^{-1}\left(\frac{10}{10}\right)=45°$

$$10\cos(\omega t)+10\sin(\omega t)\underbrace{=}_{\text{by }(4.75)}\sqrt{200}\cos(\omega t-45°)$$

(ii) Using (4.75) with $a=\sqrt{2}$ and $b=2$ we have

$$R=\sqrt{2^2+\left(\sqrt{2}\right)^2}=\sqrt{6}$$

$$\beta=\tan^{-1}\left(\frac{2}{\sqrt{2}}\right)=54.74°$$

$$\sqrt{2}\cos(\omega t)+2\sin(\omega t)=\sqrt{6}\cos(\omega t-54.74°)$$

(iii) We have $-2\sin(\omega t)+5\cos(\omega t)$. Using (4.75) with $a=5$ and $b=-2$:

$$R=\sqrt{(-2)^2+5^2}=\sqrt{29} \text{ and } \beta=\tan^{-1}\left(-\frac{2}{5}\right)=-21.80°$$

$$-2\sin(\omega t)+5\cos(\omega t)=\sqrt{29}\cos(\omega t+21.80°)$$

(iv) We have $R=\sqrt{2}$, $\beta=\tan^{-1}\left(-\frac{1}{1}\right)=-45°$. Therefore

$$\cos(\omega t)-\sin(\omega t)=\sqrt{2}\cos(\omega t+45°)$$

(v) We have $R=2$, the angle $=\tan^{-1}\left(\sqrt{3}\right)=60°$, but $2\cos(60°)$ is positive and a is negative so angle $\beta=60°+180°=240°$:

$$-\cos(\omega t)-\sqrt{3}\sin(\omega t)=2\cos(\omega t-240°)$$

(vi) $R=\sqrt{50}$, angle $=\tan^{-1}(-1)=-45°$, but $\cos(-45°)$ is positive and a is negative so angle $\beta=-45°+180°=135°$:

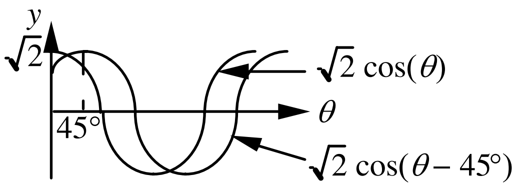
$$-5\cos(\omega t)+5\sin(\omega t)=\sqrt{50}\cos(\omega t-135°)$$

2. We first convert each function into the form $R\cos(\theta\pm\beta)$ by using (4.75): (i) We have

$$R=\sqrt{2} \text{ and } \beta=\tan^{-1}(1)=45°$$

$$\sin(\theta)+\cos(\theta)=\sqrt{2}\cos(\theta-45°)$$

$\sqrt{2}\cos(\theta-45°)$ is the graph of $\sqrt{2}\cos(\theta)$ but lagging by $45°$:



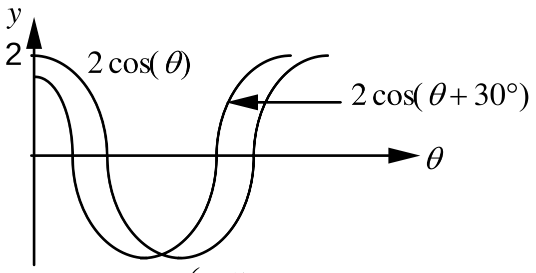
(ii) Applying (4.75) with $a=\sqrt{3}$ and $b=-1$ gives:

$$R=2 \text{ and } \beta=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-30°$$

$$\sqrt{3}\cos(\theta)-\sin(\theta)=2\cos(\theta+30°)$$

(4.75) $a\cos(\theta)+b\sin(\theta)=R\cos(\theta-\beta)$ with $R=\sqrt{a^2+b^2}$ and $\beta=\tan^{-1}\left(\frac{b}{a}\right)$

$2\cos(\theta + 30°)$ is the graph of $2\cos(\theta)$ but leading by $30°$:

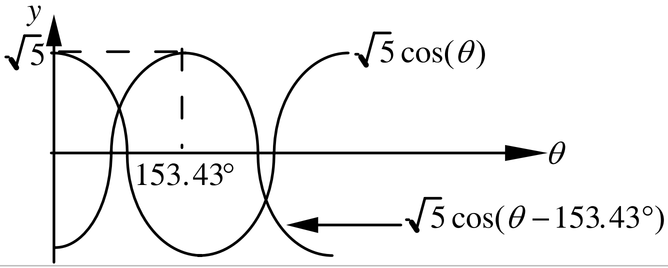


(iii) Similarly $R = \sqrt{5}$ and $\beta = \tan^{-1}\left(-\frac{1}{2}\right) = -26.57°$.

We have $\cos(-26.57)$ is positive but $a = -2$. Hence
$\beta = -26.57° + 180° = 153.43°$

$$\sin(\theta) - 2\cos(\theta) = \sqrt{5}\cos(\theta - 153.43°)$$

$\sqrt{5}\cos(\theta - 153.43°)$ is the graph of $\sqrt{5}\cos(\theta)$ but lagging by $153.43°$:



3. Applying (4.75) with $a = 3$ and $b = \sqrt{3}$ we have:

$$R = \sqrt{3^2 + \left(\sqrt{3}\right)^2} = \sqrt{12} \text{ and } \beta = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -60° = -\frac{\pi}{3}\,rad$$

Hence

$$i = 3\cos(3t) - \sqrt{3}\sin(3t) = \sqrt{12}\cos\left(3t + \frac{\pi}{3}\right)$$

Amplitude $= \sqrt{12}$ and period $= \frac{2\pi}{3}$ because $\omega = 3$.

4. Using $l = 1$, $g = 10$ we have $\omega = \sqrt{\frac{10}{1}} = \sqrt{10}$.

Substituting $A = 1$, $B = 0.5$ and $\omega = \sqrt{10}$ into $\theta = A\cos(\omega t) + B\sin(\omega t)$ gives:

$$\theta = \cos\left(\sqrt{10}t\right) + 0.5\sin\left(\sqrt{10}t\right)$$

We use (4.75) to put θ into $R\cos(\omega t \pm \beta)$ form with $a = 1$, $b = 0.5$:

$$R = \sqrt{1^2 + 0.5^2} = \sqrt{1.25} = 1.12,\ \beta = \tan^{-1}\left(\frac{0.5}{1}\right) = 26.57°$$

Hence $\theta = 1.12\cos\left(\sqrt{10}t - 26.57°\right)$. Amplitude $= 1.12$, period $= 2\pi/\sqrt{10} = 1.99s$ and phase $= 26.57°$ lagging.

(4.76) $a\cos(\theta) + b\sin(\theta) = R\cos(\theta - \beta)$ with $R = \sqrt{a^2 + b^2}$ and $\beta = \tan^{-1}\left(\frac{b}{a}\right)$

5. (i) By substituting $A=\sqrt{3}$, $B=1$ and $\omega=10$ into $x=A\cos(\omega t)+B\sin(\omega t)$ we have:

$$x=\sqrt{3}\cos(10t)+\sin(10t)$$

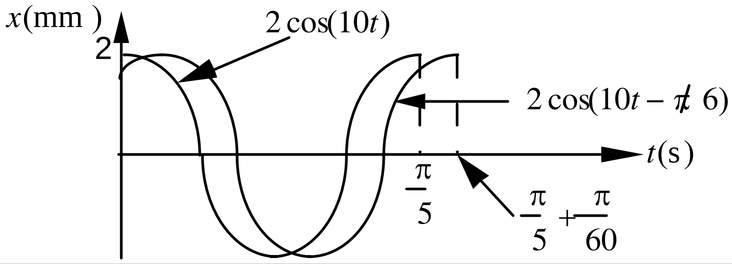
Applying (4.75) with $a=\sqrt{3}$ and $b=1$ gives:

$$R=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2 \text{ and } \beta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

Hence $x=2\cos\left(10t-\frac{\pi}{6}\right)$. So the amplitude = 2.

(ii) For sketch we need to find the period and time displacement of $x=2\cos\left(10t-\frac{\pi}{6}\right)$. The period $T=\frac{2\pi}{10}=\frac{\pi}{5}\text{s}$ and time displacement $=\frac{\pi/6}{10}=\frac{\pi}{60}\text{s}$.

So $x=2\cos\left(10t-\frac{\pi}{6}\right)$ lags $2\cos(10t)$ by $\frac{\pi}{60}$ s.

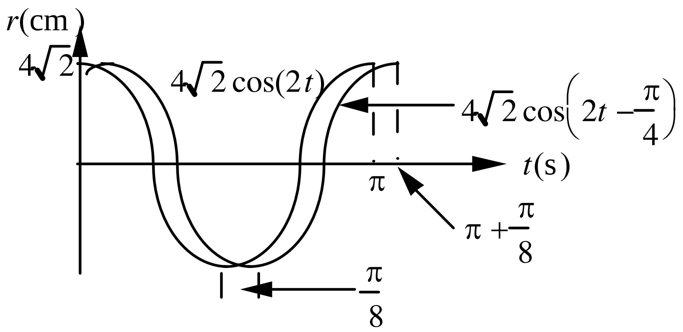


6. Using (4.75) we have $r=4\sqrt{2}\cos\left(2t-\frac{\pi}{4}\right)$.

Amplitude $=4\sqrt{2}$, period $=\frac{2\pi}{2}=\pi\text{ s}$. For sketch we have to evaluate the time displacement:

$$\text{time displacement} \underbrace{=}_{\text{by (4.33)}} \frac{\pi/4}{2}=\frac{\pi}{8}\text{s}$$

$r=4\sqrt{2}\cos\left(2t-\frac{\pi}{4}\right)$ lags $4\sqrt{2}\cos(2t)$ by $\frac{\pi}{8}$s.



(4.33) time displacement$=\frac{\alpha}{\omega}$

(4.76) $a\cos(\theta)+b\sin(\theta)=R\cos(\theta-\beta)$ with $R=\sqrt{a^2+b^2}$ and $\beta=\tan^{-1}\left(\frac{b}{a}\right)$