Complete solutions to Exercise 6(d)

1. Similar to **EXAMPLE 20**. We have:

$$\frac{d}{dt}\left[5te^{-5t}\right]=\underbrace{\left[5e^{-5t}-25te^{-5t}\right]}_{\text{by (6.31)}}$$
$$=5e^{-5t}\left[1-5t\right] \quad \text{(Factorizing)}$$

2. Substituting $C=1\times 10^{-6}$ and $v=(t+1)e^{-1000t}$ $\left(\text{remember } 1\times10^{3}=1000\right)$ into i gives:

$$i=\left(1\times10^{-6}\right)\frac{d}{dt}\left[\left(t+1\right)e^{-1000t}\right]$$
$$=\left(1\times10^{-6}\right)\underbrace{\left[e^{-1000t}-1000\left(t+1\right)e^{-1000t}\right]}_{\text{by (6.31)}}$$
$$=\left(1\times10^{-6}\right)\left[1-1000t-1000\right]e^{-1000t} \quad \text{(Opening Brackets)}$$
$$=-\left(1\times10^{-6}\right)\left[999+1000t\right]e^{-1000t}$$

3. (a) $\frac{d}{dt}\left[t\sin(t)\right]\underbrace{=}_{\text{by (6.31)}}1.\sin(t)+t\cos(t)=\sin(t)+t\cos(t)$

(b) Let $u=x$, $\frac{du}{dx}=1$ and $v=\ln(x)$, $\frac{dv}{dx}=\frac{1}{x}$. Substituting these into (6.31) gives:

$$\frac{d}{dx}\left[x\ln(x)\right]=1.\ln(x)+x.\frac{1}{x}$$
$$=\ln(x)+1$$

(c) Let $u=\theta$, $u'=1$ and $v=e^{\theta}$, $v'=e^{\theta}$. Using (6.31) gives

$$\frac{d}{d\theta}\left(\theta e^{\theta}\right)=1.e^{\theta}+\theta e^{\theta}=e^{\theta}\left(1+\theta\right) \quad \text{[Factorizing]}$$

(d) Using (6.31)

$$\frac{d}{du}\left[e^{u}\sin(u)\right]=e^{u}\sin(u)+e^{u}\cos(u)$$
$$=e^{u}\left[\sin(u)+\cos(u)\right] \quad \text{(Factorizing)}$$

(e) By using (4.53) we have

$$\sin(\alpha)\cos(\alpha)=\frac{1}{2}\sin(2\alpha)$$

So

$$\frac{d}{d\alpha}\left[\sin(\alpha)\cos(\alpha)\right]=\frac{d}{d\alpha}\left[\frac{1}{2}\sin(2\alpha)\right]$$
$$=\frac{1}{2}\frac{d}{d\alpha}\left[\sin(2\alpha)\right]=\frac{1}{2}\left[2\cos(2\alpha)\right]=\cos(2\alpha)$$

(f) Let $u=q$ and $v=1+q$. Then $\frac{du}{dq}=1$ and $\frac{dv}{dq}=1$.

(4.53) $\quad 2\sin(A)\cos(A)=\sin(2A)$

(6.31) $\quad (uv)'=u'v+uv'$

Substituting these into (6.32) gives:

$$\frac{d}{dq}\left(\frac{q}{1+q}\right)=\frac{1.(1+q)-q.1}{(1+q)^2}$$

$$=\frac{1+q-q}{(1+q)^2}$$

$$=\frac{1}{(1+q)^2}$$

(g) Let $u=\cos(a)$, $u'=-\sin(a)$ and $v=1+\sin(a)$, $v'=\cos(a)$

Substituting these into (6.32) gives

$$\frac{d}{da}\left[\frac{\cos(a)}{1+\sin(a)}\right]=\frac{-\sin(a).\left[1+\sin(a)\right]-\cos(a).\cos(a)}{\left[1+\sin(a)\right]^2}$$

$$=\frac{-\sin(a)-\sin^2(a)-\cos^2(a)}{\left[1+\sin(a)\right]^2}$$

$$=\frac{-\sin(a)-\left[\sin^2(a)+\cos^2(a)\right]}{\left[1+\sin(a)\right]^2}$$

$$=\frac{-\sin(a)-1}{\left[1+\sin(a)\right]^2}=\frac{-\left[1+\sin(a)\right]}{\left[1+\sin(a)\right]^2}=-\frac{1}{1+\sin(a)}\quad\text{(Cancelling)}$$

(h) Let $u=e^z$, $u'=e^z$ and $v=\cos(z)$, $v'\underbrace{=}_{\text{by (6.7)}}-\sin(z)$:

Substituting these into (6.32) gives

$$\frac{d}{dz}\left[\frac{e^z}{\cos(z)}\right]=\frac{e^z\cos(z)-e^z.\left[-\sin(z)\right]}{\left[\cos(z)\right]^2}$$

$$=\frac{e^z\left[\cos(z)+\sin(z)\right]}{\cos^2(z)}$$

(i) Let $u=M^3$, $u'=3M^2$ and $v=\ln(M)$, $v'\underbrace{=}_{\text{by (6.9)}}\frac{1}{M}$

Substituting into (6.31):

$$\frac{d}{dM}\left[M^3\ln(M)\right]=3M^2\ln(M)+M^3\frac{1}{M}$$

$$=3M^2\ln(M)+M^2=M^2\left(3\ln(M)+1\right)\quad\left[\text{Factorizing}\right]$$

(6.7) $\left[\cos(x)\right]'=-\sin(x)$

(6.9) $\left[\ln(x)\right]'=\frac{1}{x}$

(6.31) $(uv)'=u'v+uv'$

(6.32) $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$

(j) Let $u=\beta^2$, $u'=2\beta$ and $v=\sin(\beta)$, $v'=\cos(\beta)$. Applying (6.32) gives:

$$\frac{d}{d\beta}\left(\frac{\beta^2}{\sin(\beta)}\right)=\frac{2\beta\sin(\beta)-\beta^2\cos(\beta)}{(\sin\beta)^2}$$

$$=\frac{\beta\left[2\sin(\beta)-\beta\cos(\beta)\right]}{\sin^2\beta}$$

(k) Let $u=h$ then $u'=1$ and $v=\sqrt{1+h^2}=\left(1+h^2\right)^{1/2}$

$$v'=\frac{1}{\cancel{2}}\left(1+h^2\right)^{-1/2}.\cancel{2}h=\frac{h}{\left(1+h^2\right)^{1/2}}$$

Using (6.32)

$$\frac{d}{dh}\left(\frac{h}{\sqrt{1+h^2}}\right)=\frac{\left(1+h^2\right)^{1/2}-\dfrac{h.h}{\left(1+h^2\right)^{1/2}}}{\left[\left(1+h^2\right)^{1/2}\right]^2}$$

$$=\frac{\left(1+h^2\right)^{1/2}\left(1+h^2\right)^{1/2}-h^2}{\left(1+h^2\right)\left(1+h^2\right)^{1/2}}\quad\left[\text{Multiplying by }\left(1+h^2\right)^{1/2}\right]$$

$$=\frac{\left(1+h^2\right)-h^2}{\left(1+h^2\right)^{3/2}}=\frac{1}{\left(1+h^2\right)^{3/2}}\quad\left[\text{Simplfying Numerator}\right]$$

(l) Let $u=J$, $u'=1$ and $v=\ln\left[\sin(J)\right]$, $v'=\dfrac{1}{\sin(J)}.\cos(J)=\dfrac{\cos(J)}{\sin(J)}=\cot(J)$

$$\frac{d}{dJ}\left\{J\ln\left[\sin(J)\right]\right\}\underbrace{=}_{\text{by (6.31)}}1.\ln\left[\sin(J)\right]+J.\cot(J)$$

$$=\ln\left[\sin(J)\right]+J\cot(J)$$

(m) Let $u=e^{\Sigma}$ then $u'=e^{\Sigma}$ and $v=\sin(2\Sigma)$ then $v'=2\cos(2\Sigma)$

Substituting $u=e^{\Sigma}$, $u'=e^{\Sigma}$, $v=\sin(2\Sigma)$ and $v'=2\cos(2\Sigma)$ into (6.31):

$$\frac{d}{d\Sigma}\left[e^{\Sigma}\sin(2\Sigma)\right]=e^{\Sigma}\sin(2\Sigma)+e^{\Sigma}.\left[2\cos(2\Sigma)\right]$$

$$=e^{\Sigma}\left[\sin(2\Sigma)+2\cos(2\Sigma)\right]\quad(\text{Factorizing})$$

(n) Let $u=(Z+1)/(Z-1)$ then

$$\frac{d}{dZ}\left[\left(\frac{Z+1}{Z-1}\right)^{1/2}\right]=\frac{d}{dZ}\left[u^{1/2}\right]=\frac{1}{2}u^{-1/2}\frac{du}{dZ}\qquad(*)$$

We need to find $\dfrac{du}{dZ}$. How?

Using the quotient rule we have $\dfrac{du}{dZ}=\dfrac{(1)\cdot(Z-1)-(Z+1)\cdot(1)}{(Z-1)^2}=\dfrac{-2}{(Z-1)^2}$

(6.31) $\quad (uv)'=u'v+uv'$

(6.32) $\quad (u/v)'=\dfrac{u'v-uv'}{v^2}$

Substituting into (*) gives:

$$\frac{d}{dZ}\left[\left(\frac{Z+1}{Z-1}\right)^{1/2}\right]=\frac{1}{\cancel{2}}u^{-1/2}\frac{-\cancel{2}}{(Z-1)^2}$$

$$=-\left(\frac{Z+1}{Z-1}\right)^{-1/2}.\frac{1}{(Z-1)^2}$$

$$=-\frac{(Z+1)^{-1/2}}{(Z-1)^{-1/2}(Z-1)^2}\quad\left[\text{Writing }\left(\frac{Z+1}{Z-1}\right)^{-1/2}=\frac{(Z+1)^{-1/2}}{(Z-1)^{-1/2}}\right]$$

$$=-\frac{1}{(Z+1)^{1/2}(Z-1)^{3/2}}\quad\left[\text{Writing }(Z+1)^{-1/2}=\frac{1}{(Z+1)^{1/2}}\right]$$

$$=-\frac{1}{\sqrt{(Z+1)}\sqrt{(Z-1)^3}}=-\sqrt{\frac{1}{(Z+1)(Z-1)^3}}$$

(o) How do we find $\frac{d}{d\Sigma}\left[\Sigma e^{\Sigma}\sin(5\Sigma)\right]$?

Need to use the product rule (6.31). Let $u=\Sigma e^{\Sigma}$ and $v=\sin(5\Sigma)$.

To use (6.31) we need to find $\frac{du}{d\Sigma}$ and $\frac{dv}{d\Sigma}$.

$$\frac{dv}{d\Sigma}\underbrace{=}_{\text{by (6.12)}}5\cos(5\Sigma)$$

How do we find $\frac{du}{d\Sigma}$?

Need to apply the product rule (6.31) to $u=\Sigma e^{\Sigma}$. Hence

$$\frac{du}{d\Sigma}=1.e^{\Sigma}+\Sigma e^{\Sigma}=e^{\Sigma}(1+\Sigma)\quad\text{[Factorizing]}$$

Using the product rule on the original function with $u=\Sigma e^{\Sigma}$, $u'=e^{\Sigma}(1+\Sigma)$, $v=\sin(5\Sigma)$ and $v'=5\cos(5\Sigma)$ gives

$$\frac{d}{d\Sigma}\left[\Sigma.e^{\Sigma}\sin(5\Sigma)\right]=e^{\Sigma}(1+\Sigma)\cdot\sin(5\Sigma)+\Sigma e^{\Sigma}\cdot 5\cos(5\Sigma)$$

$$=\underbrace{e^{\Sigma}}_{\text{factorizing }e^{\Sigma}}\left[(1+\Sigma)\sin(5\Sigma)+5\Sigma\cos(5\Sigma)\right]$$

(p) $\ln\left(\frac{\delta^2+1}{\delta^2-1}\right)\underbrace{=}_{\text{by (5.12)}}\ln(\delta^2+1)-\ln(\delta^2-1)$

(5.12) $\ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)$

(6.12) $\left[\sin(kx)\right]'=k\cos(kx)$

(6.31) $(uv)'=u'v+uv'$

$$\frac{d}{d\delta}\left[\ln\left(\delta^2+1\right)-\ln\left(\delta^2-1\right)\right]=\frac{1}{\delta^2+1}.2\delta-\frac{1}{\delta^2-1}.2\delta$$

$$=\frac{2\delta}{\delta^2+1}-\frac{2\delta}{\delta^2-1}$$

$$=\frac{2\delta\left(\delta^2-1\right)-2\delta\left(\delta^2+1\right)}{\left(\delta^2+1\right)\left(\delta^2-1\right)}$$

$$=\frac{2\delta^3-2\delta-2\delta^3-2\delta}{\delta^4-1}\quad\left[\text{Expanding Brackets}\right]$$

$$=\frac{-4\delta}{\delta^4-1}\quad\left[\text{Simplifying Numerator}\right]$$

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4. Let $u=e^{-kt}$ $\qquad v=\cos(\omega t)$

$u'=-ke^{-kt}$ $\qquad v=-\omega\sin(\omega t)$

$$\frac{ds}{dt}=-ke^{-kt}\cos\left(\omega t\right)-e^{-kt}\omega\sin\left(\omega t\right)$$

$$=-e^{-kt}\left[k\cos\left(\omega t\right)+\omega\sin\left(\omega t\right)\right]\quad\left(\text{Factorizing}\right)$$
