Complete solutions to Exercise 6(e)

1. (a)

$$x = t^3 - 10t^2 + 40t - 30,\ \frac{dx}{dt} = 3t^2 - 20t + 40,\ \frac{d^2x}{dt^2} = 6t - 20$$

(b)

$$x = 7t^3 - 50t^2 + 50t,\ \frac{dx}{dt} = 21t^2 - 100t + 50,\ \frac{d^2x}{dt^2} = 42t - 100$$

(c)

$$x = 5\cos(2t),\ \frac{dx}{dt} \underbrace{=}_{\text{by (6.13)}} -10\sin(2t),\ \frac{d^2x}{dt^2} \underbrace{=}_{\text{by (6.12)}} -20\cos(2t)$$

(d)

$$x = 3\cos(10\pi t) + 5\sin(10\pi t)$$

$$\frac{dx}{dt} = \underbrace{(-3\times 10\pi)\sin(10\pi t)}_{\text{by (6.13)}} + \underbrace{(5\times 10\pi)\cos(10\pi t)}_{\text{by (6.12)}} = -30\pi\sin(10\pi t) + 50\pi\cos(10\pi t)$$

$$\frac{d^2x}{dt^2} = (-3\times 10\pi\times 10\pi)\cos(10\pi t) + [5\times 10\pi\times(-10\pi)]\sin(10\pi t)$$

$$= -300\pi^2\cos(10\pi t) - 500\pi^2\sin(10\pi t) \quad [\text{Simplifying}]$$

$$= -100\pi^2[3\cos(10\pi t) + 5\sin(10\pi t)] \quad (\text{Factorizing})$$

2. (i) We have

$$x = 2t^3 - 15t^2 - 36t$$

$$v = \dot{x} = 6t^2 - 30t - 36 \quad [\text{Differentiating}]$$

$$a = \ddot{x} = 12t - 30 \quad [\text{Differentiating}]$$

(ii) The particle comes to rest at $v = 0$.

$$6t^2 - 30t - 36 = 0$$

$$t^2 - 5t - 6 = 0 \quad \text{(Dividing through by 6)}$$

$$(t-6)(t+1) = 0$$

$$t = 6,\ t = -1$$

Hence the particle comes to rest after 6 seconds.

3. We have

$$h = 10t - 4.9t^2,\ \dot{h} = 10 - 9.8t,\ \ddot{h} = -9.8$$

Hence $a = -9.8m/s^2$. This means that the acceleration is constant for all t.

4.

$$h = 4.9t^2,\ \dot{h} = 9.8t,\ \ddot{h} = 9.8$$

Substituting $t = 1.5$,

$$v = \dot{h} = 9.8\times 1.5 = 14.7m/s \quad \text{and} \quad a = \ddot{h} = 9.8m/s^2$$

(6.12) $[\sin(kt)]' = k\cos(kt)$

(6.13) $[\cos(kt)]' = -k\sin(kt)$

5. (a) We have

$$x = 3t - \sin(3t),\ \dot{x} = 3 - 3\cos(3t),\ \ddot{x} = 0 - (-3\times 3)\sin(3t) = 9\sin(3t)$$

(b)

$$x = \frac{1 - e^{-t/5}}{5} = \frac{1}{5} - \frac{e^{-t/5}}{5},\ \dot{x} = 0 - \left(-\frac{1}{5}\right)\frac{e^{-t/5}}{5} = \frac{e^{-t/5}}{25},\ \ddot{x} = -\frac{1}{5}\frac{e^{-t/5}}{25} = -\frac{e^{-t/5}}{125}$$

(c) $x = te^{-t}$. To differentiate x we need to use the product rule (6.31).

$$u = t \qquad v = e^{-t}$$

$$u' = 1 \qquad v' = -e^{-t}$$

$$\dot{x} = (1)e^{-t} + t\left(-e^{-t}\right) = e^{-t} - te^{-t} = (1-t)e^{-t}$$

To find the acceleration, a, differentiate $\dot{x}$ again:

$$\ddot{x} = -e^{-t} - \underbrace{\left[e^{-t} - te^{-t}\right]}_{\text{This is just } \dot{x}} = -2e^{-t} + te^{-t} = (t-2)e^{-t}$$

(d)

$$x = \ln\left(1 + 3t^2\right)$$

$$\dot{x} \underbrace{=}_{\text{by (6.18)}} \frac{1}{1+3t^2}.6t = \frac{6t}{1+3t^2}$$

How do we find $\ddot{x}$?

Need to use the quotient rule (6.32).

$$u = 6t \qquad v = 1 + 3t^2$$

$$u' = 6 \qquad v' = 6t$$

$$\ddot{x} = \frac{6\left(1+3t^2\right) - 6t.6t}{\left(1+3t^2\right)^2} = \frac{6 + 18t^2 - 36t^2}{\left(1+3t^2\right)^2} = \frac{6 - 18t^2}{\left(1+3t^2\right)^2}$$

(e) $x = 0.5\ln\left[\cosh\left(\sqrt{20}t\right)\right]$

$$\dot{x} \underbrace{=}_{\text{by (6.18)}} 0.5.\frac{1}{\cosh\left(\sqrt{20}t\right)}.\frac{d}{dt}\left[\cosh\left(\sqrt{20}t\right)\right]$$

$$= \frac{0.5}{\cosh\sqrt{20}t}.\underbrace{\sqrt{20}\sinh\left(\sqrt{20}t\right)}_{\text{by (6.26)}} = 0.5.\sqrt{20}\frac{\sinh\left(\sqrt{20}t\right)}{\cosh\left(\sqrt{20}t\right)} = \frac{1}{2}.\sqrt{20}\tanh\left(\sqrt{20}t\right) = \sqrt{5}\tanh\left(\sqrt{20}t\right)$$

$$\ddot{x} \underbrace{=}_{\text{by (6.27)}} \sqrt{5}\text{sech}^2\left(\sqrt{20}t\right).\sqrt{20} = \sqrt{100}\text{sech}^2\left(\sqrt{20}t\right) = 10\text{sech}^2\left(\sqrt{20}t\right)$$

(6.18) $\left[\ln(u)\right]' = \frac{1}{u}\frac{du}{dt}$

(6.26) $\left[\cosh(u)\right]' = \sinh(u)\frac{du}{dt}$

(6.27) $\left[\tanh(u)\right]' = \sec h^2(u)\frac{du}{dt}$

(6.31) $(uv)' = u'v + uv'$ (6.32) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

6. **EXAMPLE 24** with $\omega = 3$, $A = 1$ and $B = -1$.
7. Very similar to **EXAMPLE 26**. $f^{v}(x) = -\sin(x)$, $f^{v}(0) = -\sin(0) = 0$
8. From $PV^n = C$ we have:

$$P = \frac{C}{V^n} = CV^{-n}$$

$$\frac{dP}{dV} = -nCV^{-n-1}$$

$$\frac{d^2P}{dV^2} = -n(-n-1)CV^{-n-2}$$

$$= n(n+1)CV^{-(n+2)} \quad \left[\text{Taking out } (-)(-) = +\right]$$

$$= \frac{n(n+1)C}{V^{n+2}} \underbrace{=}_{\substack{\text{substituting}\\ C=PV^n}} \frac{n(n+1)PV^n}{V^nV^2} = \frac{n(n+1)P}{V^2} \quad (\text{Cancelling } V^n\text{'s})$$

9. To find $\dot{x}$ we need to use the product rule, (6.31), on $x = (1+t)e^{-\omega t}$.

$$u = 1+t \qquad v = e^{-\omega t}$$

$$u' = 1 \qquad v' = -\omega e^{-\omega t}$$

$$\dot{x} = (1)e^{-\omega t} + (1+t)\left(-\omega e^{-\omega t}\right) = (1-\omega-\omega t)e^{-\omega t}$$

To find $\ddot{x}$ we need to apply (6.31) again:

$$u = 1-\omega-\omega t \qquad v = e^{-\omega t}$$

$$u' = -\omega \qquad v' = -\omega e^{-\omega t}$$

$$\ddot{x} = (-\omega).e^{-\omega t} + (1-\omega-\omega t)\left(-\omega e^{-\omega t}\right) = \left(-\omega-\omega+\omega^2+\omega^2 t\right)e^{-\omega t}$$

$$\ddot{x} = \left(\omega^2 t + \omega^2 - 2\omega\right)e^{-\omega t}$$

Substituting for $\ddot{x}$, $\dot{x}$ and x into $\ddot{x} + 2\omega\dot{x} + \omega^2 x$ gives:

$$\underbrace{\left(\omega^2 t + \omega^2 - 2\omega\right)e^{-\omega t}}_{=\ddot{x}} + 2\omega\underbrace{(1-\omega-\omega t)e^{-\omega t}}_{=\dot{x}} + \omega^2\underbrace{(1+t)e^{-\omega t}}_{=x}$$

$$= \left(\omega^2 t + \omega^2 - 2\omega + 2\omega - 2\omega^2 - 2\omega^2 t + \omega^2 + \omega^2 t\right)e^{-\omega t} = 0$$

10. We have $u = A\sin\left(\frac{\omega x}{k}\right) + B\cos\left(\frac{\omega x}{k}\right)$.

$$\frac{du}{dx} = \underbrace{\left(\frac{\omega}{k}\right)A\cos\left(\frac{\omega x}{k}\right)}_{\text{by } (6.12)} - \underbrace{\left(\frac{\omega}{k}\right)B\sin\left(\frac{\omega x}{k}\right)}_{\text{by } (6.13)}$$

$$\frac{d^2u}{dx^2} = -\underbrace{\left(\frac{\omega}{k}\right)\left(\frac{\omega}{k}\right)A\sin\left(\frac{\omega x}{k}\right)}_{\text{by } (6.13)} - \underbrace{\left(\frac{\omega}{k}\right)\left(\frac{\omega}{k}\right)B\cos\left(\frac{\omega x}{k}\right)}_{\text{by } (6.12)} = -\frac{\omega^2}{k^2}\underbrace{\left[A\sin\left(\frac{\omega x}{k}\right) + B\cos\left(\frac{\omega x}{k}\right)\right]}_{=u}$$

$$\frac{d^2u}{dx^2} = -\frac{\omega^2}{k^2}u$$

(6.12) $\left[\sin(kx)\right]' = k\cos(kx)$

(6.13) $\left[\cos(kx)\right]' = -k\sin(kx)$

(6.31) $(uv)' = u'v + uv'$