Complete solutions to Exercise 6(f)

1.(a) We have

$$x = \sin(t), \quad \frac{dx}{dt} = \cos(t)$$

$$y = \cos(t), \quad \frac{dy}{dt} = -\sin(t)$$

By (6.36):

$$\frac{dy}{dx} = \frac{-\sin(t)}{\cos(t)} = -\tan(t) \quad \left[\text{Because } \frac{\sin}{\cos} = \tan\right]$$

(b)

$$x = 2t^3 - t^2, \quad \frac{dx}{dt} = 6t^2 - 2t$$

$$y = 10t^2 - t^3, \quad \frac{dy}{dt} = 20t - 3t^2$$

By (6.36)

$$\frac{dy}{dx} = \frac{20t - 3t^2}{6t^2 - 2t} = \frac{\not{t}(20 - 3t)}{\not{t}(6t - 2)} = \frac{20 - 3t}{6t - 2} \quad [\text{Cancelling } t\text{'s}]$$

(c)

$$x = (t-3)^2, \quad \frac{dx}{dt} = 2(t-3)$$

$$y = t^3 - 1, \quad \frac{dy}{dt} = 3t^2$$

By (6.36), $\dfrac{dy}{dx} = \dfrac{3t^2}{2(t-3)}$.

(d)

$$x = e^t - 1, \quad \frac{dx}{dt} = e^t$$

$$y = e^{t/2}, \quad \frac{dy}{dt} = \frac{1}{2}e^{t/2} \quad \left[\text{Because } \left(e^{kt}\right)' = ke^{kt}\right]$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}e^{t/2}}{e^t} = \frac{1}{2}e^{t/2-t} = \frac{1}{2}e^{-t/2} = \frac{1}{2e^{t/2}}$$

2.(a) The MAPLE output is the following:

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> plot([t*sin(t),t*cos(t),t=0..50]);
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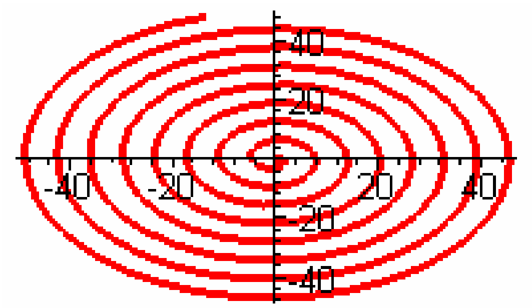

(6.36) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

2.(b) How do we differentiate $x = t\sin(t)$?
Using the product rule (6.31):

$$u = t \qquad v = \sin(t)$$
$$u' = 1 \qquad v' = \cos(t)$$

$$\frac{dx}{dt} = 1.\sin(t) + t.\cos(t) = \sin(t) + t.\cos(t)$$

We have $y = t\cos(t)$. Similarly for $\frac{dy}{dt}$;

$$u = t \qquad v = \cos(t)$$
$$u' = 1 \qquad v' = -\sin(t)$$
$$\frac{dy}{dt} = 1.\cos(t) + t\left[-\sin(t)\right] = \cos(t) - t\sin(t)$$

By (6.36)

$$\frac{dy}{dx} = \frac{\cos(t) - t\sin(t)}{\sin(t) + t\cos(t)}$$

3.(i) The Maple output.

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> plot([(t-2)^2,t^3-5,t=0..3.5]);
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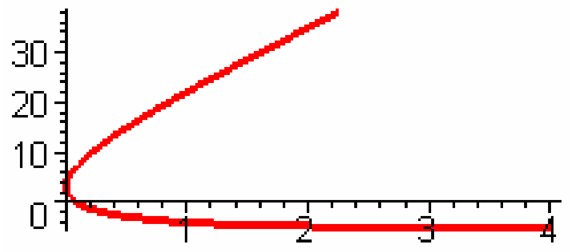

(ii)

$$x = (t-2)^2, \quad v_x = \dot{x} = 2(t-2)$$
$$y = t^3 - 5, \quad v_y = \dot{y} = 3t^2$$

Hence

$$v = \sqrt{[2(t-2)]^2 + (3t^2)^2} = \sqrt{4(t^2 - 4t + 4) + 9t^4} = \sqrt{9t^4 + 4t^2 - 16t + 16}$$

4. (i) Maple commands are:

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> plot([(t^2+2)^(3/2)/3,t,t=0..5]);
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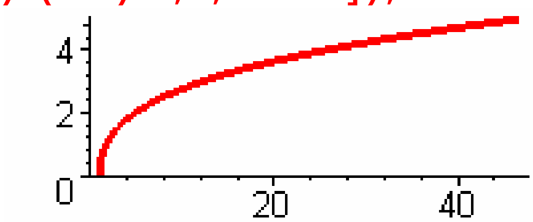

(ii) $x = \frac{(t^2+2)^{3/2}}{3}$. Differentiating,

(6.31) $\quad (uv)' = u'v + uv'$

(6.36) $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$v_x = \dot{x} = \frac{3\left(t^2+2\right)^{1/2}}{2\times 3}.2t = \left(t^2+2\right)^{1/2}.t \quad \text{[Cancelling]}$$

$$y = t,\ v_y = \dot{y} = 1$$

(iii) $v = \sqrt{\left[\left(t^2+2\right)^{1/2} t\right]^2 + 1^2} = \sqrt{\left(t^2+2\right)t^2+1}$

$$= \sqrt{t^4+2t^2+1} = \sqrt{\left(t^2+1\right)^2} = t^2+1$$

Hence $v = t^2 + 1$.

5. (i) Opening up the brackets gives $x = 16t - 8t^3,\ v_x = \dot{x} = 16 - 24t^2$.

We have $y = t^3,\ v_y = \dot{y} = 3t^2$.

(ii)

$$v = \sqrt{\left(16-24t^2\right)^2 + \left(3t^2\right)^2}$$

$$= \sqrt{256 - 768t^2 + 576t^4 + 9t^4}$$

$$= \sqrt{256 - 768t^2 + 585t^4}$$

6. We have

$$x = r\cos(t),\ v_x = \dot{x} = -r\sin(t),\ a_x = \ddot{x} = -r\cos(t)$$

$$y = r\sin(t),\ v_y = \dot{y} = r\cos(t),\ a_y = \ddot{y} = -r\sin(t)$$

$$v = \sqrt{\left[-r\sin(t)\right]^2 + \left[r\cos(t)\right]^2} = \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} = \sqrt{r^2\underbrace{\left[\sin^2(t)+\cos^2(t)\right]}_{=1}} = r$$

$$a = \sqrt{\left[-r\cos(t)\right]^2 + \left[-r\sin(t)\right]^2} = \sqrt{r^2\cos^2(t) + r^2\sin^2(t)} = r$$

7. (i) We have $x = t^3 - t^2$:

$$v_x = \dot{x} = 3t^2 - 2t,\ a_x = \ddot{x} = 6t - 2$$

Also $y = 5t^2 - t^3$:

$$v_y = \dot{y} = 10t - 3t^2,\ a_y = \ddot{y} = 10 - 6t$$

(ii) Substituting $t = 2$, $v_x = \left(3\times 2^2\right) - \left(2\times 2\right) = 8$, $v_y = \left(10\times 2\right) - \left(3\times 2^2\right) = 8$

$$v = \sqrt{8^2+8^2} = 11.31\text{m/s}$$

For the acceleration magnitude at $t = 2$ we substitute $t = 2$ into a_x and a_y.

$$a_x = \left(6\times 2\right) - 2 = 10$$

$$a_y = 10 - \left(6\times 2\right) = -2$$

$$a = \sqrt{10^2 + (-2)^2} = 10.20\text{m/s}^2$$

8. From $x = \cosh(t)$ we have $\frac{dx}{dt} = \sinh(t)$ and $y = \sinh(t)$, $\frac{dy}{dt} = \cosh(t)$.

By (6.36) $\frac{dy}{dx}=\frac{\cosh(t)}{\sinh(t)}=\coth(t)$

By (6.37)

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\coth(t)\right]}{\sinh(t)}=\frac{-\text{cosech}^2(t)}{\sinh(t)}$$

$$=-\text{cosech}^2(t).\frac{1}{\sinh(t)}=-\text{cosech}^2(t).\text{cosech}(t)=-\text{cosech}^3(t)$$

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9. We have:

$$x=r\left[\theta-\sin(\theta)\right],\ \frac{dx}{d\theta}=r\left[1-\cos(\theta)\right]$$

$$y=r\left[1-\cos(\theta)\right],\ \frac{dy}{d\theta}=r\sin(\theta)$$

By (6.36)

$$\frac{dy}{dx}=\frac{r\sin(\theta)}{r\left[1-\cos(\theta)\right]}=\frac{\sin(\theta)}{1-\cos(\theta)}\quad\left[\text{Cancelling } r's\right]$$

By (6.37)

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{d\theta}\left[\frac{\sin(\theta)}{1-\cos(\theta)}\right]}{r\left[1-\cos(\theta)\right]}\qquad(\dagger)$$

How do we find $\frac{d}{d\theta}\left[\frac{\sin(\theta)}{1-\cos(\theta)}\right]$*?*

(6.36) $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$

(6.37) $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$

Use the quotient rule (6.32).We have

$$u = \sin(\theta) \qquad v = 1 - \cos(\theta)$$

$$u' = \cos(\theta) \qquad v' = \sin(\theta)$$

$$\frac{d}{d\theta}\left[\frac{\sin(\theta)}{1-\cos(\theta)}\right] = \frac{\cos(\theta)\left[1-\cos(\theta)\right]-\sin(\theta).\sin(\theta)}{\left[1-\cos(\theta)\right]^2}$$

$$= \frac{\cos(\theta)-\cos^2(\theta)-\sin^2(\theta)}{\left[1-\cos(\theta)\right]^2}$$

$$= \frac{\cos(\theta)-\left[\cos^2(\theta)+\sin^2(\theta)\right]}{\left[1-\cos(\theta)\right]^2}$$

$$= \frac{\cos(\theta)-1}{\left[1-\cos(\theta)\right]^2} = \frac{-\left[1-\cos(\theta)\right]}{\left[1-\cos(\theta)\right]^2} = \frac{-1}{1-\cos(\theta)} \quad (\text{Cancelling})$$

Substituting into $(\dagger)$ gives $\dfrac{d^2y}{dx^2} = \dfrac{-1/\left[1-\cos(\theta)\right]}{r\left[1-\cos(\theta)\right]} = \dfrac{-1}{r\left[1-\cos(\theta)\right]^2}$

(6.32) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v-uv'}{v^2}$