Complete solutions to Exercise 7(a)

1. The stationary points are evaluated by first solving $\frac{dy}{dx} = 0$.

(a) For $y = x^2 + 3x + 1$, $\frac{dy}{dx} = 2x + 3$.

For stationary points, $2x + 3 = 0$ which gives $x = -3/2$.

How do we find the value of y at $x = -3/2$?

Substituting $x = -3/2$ into $y = x^2 + 3x + 1$:

$$y = (-3/2)^2 + [3 \times -3/2] + 1 = -5/4$$

Hence $(-3/2, -5/4)$ is a stationary point.

(b) Similarly for $y = x^3 - 3x$, $\frac{dy}{dx} = 3x^2 - 3$. For stationary points:

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0 \qquad [\text{Factorizing}]$$

$$3(x-1)(x+1) = 0 \text{ gives } x = 1,\ x = -1$$

At $x = 1,\ y = 1^3 - (3 \times 1) = -2$.

At $x = -1,\ y = (-1)^3 - [3 \times (-1)] = 2$.

Thus $(1, -2)$ and $(-1, 2)$ are stationary points of $y = x^3 - 3x$.

(c) Similar to (a) and (b). $(1, -2/3)$ and $(-1, 2/3)$ are stationary points of $y = \frac{x^3}{3} - x$.

(d) Differentiating $y = 50x^2 - x^4$ with respect to x:

$$\frac{dy}{dx} = 100x - 4x^3$$

For stationary points:

$$100x - 4x^3 = 0$$

$$25x - x^3 = 0 \qquad [\text{Divide by } 4]$$

How can we find x?

Take out a common factor of x:

$$x(25 - x^2) = 0$$

$$x = 0,\ 25 - x^2 = 0$$

$$x^2 = 25$$

$$x = 5, x = -5 \qquad [\text{Square Root}]$$

We have 3 values of x: $x = 0,\ x = 5,\ x = -5$.

At $x = 0, y = (50 \times 0^2) - 0^4 = 0$. At $x = 5, y = (50 \times 5^2) - 5^4 = 625$.

At $x = -5,\ y = 625$.

Hence $(0,0)$, $(5,625)$ and $(-5,625)$ are stationary points of $y = 50x^2 - x^4$.

2. The graph crosses the x axis where $y = 0$. This means where the numerator is zero:

$$10x - x^2 = 0,\ x(10 - x) = 0 \text{ gives } x = 0,\ x = 10$$

We can find the stationary points and their nature by using (7.1).

$$\frac{dy}{dx} = \frac{10 - 2x}{10} = 0,\ 10 - 2x = 0 \text{ gives } x = 5$$

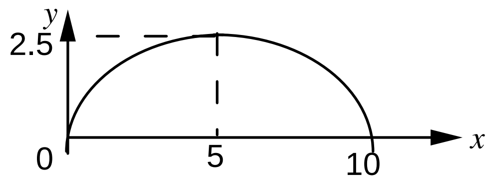
Substituting $x = 5$ into $y = \dfrac{10x - x^2}{10}$ gives

$$y = \frac{(10 \times 5) - 5^2}{10} = \frac{25}{10} = 2.5$$

Thus $(5, 2.5)$ is a stationary point of y. To find the nature we need to differentiate again

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{10 - 2x}{10}\right) = -\frac{2}{10} < 0$$

By (7.2) the stationary point $(5, 2.5)$ is a local maximum of y. Combining the above we have:



Note: This problem can also be solved by completing the square.

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3. *How do we find the maximum, minimum of* $y = x^3 + x^2 - 5x$ *?*

First we find the stationary points and then we decide the nature by using the second derivative test. For stationary points, $\dfrac{dy}{dx} = 0$.

$$y = x^3 + x^2 - 5x$$

$$\frac{dy}{dx} = 3x^2 + 2x - 5 = 0$$

Factorizing:

$$(3x + 5)(x - 1) = 0$$

$$x = -5/3,\ x = 1$$

Differentiating again $\dfrac{d^2y}{dx^2} = 6x + 2$.

Substituting $x = -5/3$, $\dfrac{d^2y}{dx^2} = 6\left(-\dfrac{5}{3}\right) + 2 < 0$. By (7.2), $x = -5/3$ is a maximum.

Substituting $x = 1$, $\dfrac{d^2y}{dx^2} > 0$. By (7.3), $x = 1$ is a minimum. *How do we find the* y *values?*

Substitute the x values $(-5/3$ and $1)$ into $y = x^3 + x^2 - 5x$:

At $x = -5/3$, $\quad y = \left(-\dfrac{5}{3}\right)^3 + \left(-\dfrac{5}{3}\right)^2 - 5\left(-\dfrac{5}{3}\right) = \dfrac{175}{27}$

At $x = 1$, $\quad y = 1^3 + 1^2 - 5 = -3$.

Thus $(-5/3,\ 175/27)$ is a local maximum and $(1, -3)$ is a local minimum.

(7.2)	$y' = 0,\ \ y'' < 0$	maximum
(7.3)	$y' = 0,\ \ y'' > 0$	minimum

4. *We first locate the stationary points, how?*

Find $\frac{ds}{dt}$ and put it to zero:

$$s = t^3 - 6t^2 + 12t$$

$$\frac{ds}{dt} = 3t^2 - 12t + 12 = 0$$

How can we solve this quadratic equation $3t^2 - 12t + 12 = 0$*?*
Factorize:

$$3\left(t^2 - 4t + 4\right) = 0$$

$$3\left(t-2\right)^2 = 0 \text{ gives } t = 2$$

To test the nature of the stationary point we use the 2nd derivative test:

$$\frac{d^2 s}{dt^2} = 6t - 12 \qquad \text{[Differentiating]}$$

Putting $t = 2$, $\frac{d^2 s}{dt^2} = (6 \times 2) - 12 = 0$
We know at $t = 2$, s does not possess a maximum or a minimum.
Does it have a horizontal point of inflexion?
We need to use (7.6):
If $t < 2$, $\frac{d^2 s}{dt^2} = 6t - 12 < 0$. If $t > 2$, $\frac{d^2 s}{dt^2} > 0$. By (7.6), the graph has a horizontal point of inflexion at $t = 2$. *What is the value of* s *?*
Substitute $t = 2$ into the original equation:

$$s = 2^3 - \left(6 \times 2^2\right) + \left(12 \times 2\right) = 8$$

Thus $(2, 8)$ is a horizontal point of inflexion.

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5. We can find where the graph cuts the t axis by putting $s = 0$ and solving the resulting equation for t.

$$t^2\left(8 - t^2\right) = 0$$

$$\text{gives } t = 0 \text{ or } 8 - t^2 = 0 \text{ so } t^2 = 8,\ t = \pm\sqrt{8}$$

The graph cuts the t axis at 0, $\sqrt{8}$ and $-\sqrt{8}$.
To find stationary points, we need to differentiate and put it to zero. Opening up the brackets:

$$s = 8t^2 - t^4$$

$$\frac{ds}{dt} = 16t - 4t^3 = 4(4t) - (4t)t^2 = 4t\left(4 - t^2\right) = 0$$

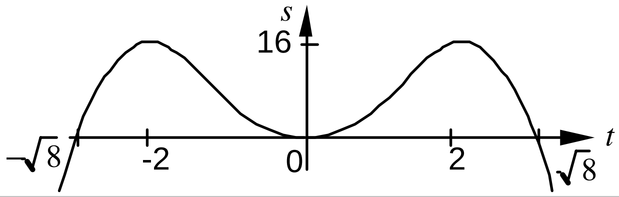
$$t = 0 \text{ or } 4 - t^2 = 0 \text{ gives } t = \pm 2$$

When $t = 0$, $s = 0$. When $t = 2$, $s = \left(8 \times 2^2\right) - 2^4 = 16$. When $t = -2$, $s = 16$.
$(0,0)$, $(2,16)$ and $(-2,16)$ are stationary points. To find the nature of these stationary points we need to find the 2nd derivative, $\frac{d^2 s}{dt^2}$.

$$\frac{d^2 s}{dt^2} = 16 - 12t^2$$

At $t = 0$, $\frac{d^2 s}{dt^2} = 16 > 0$. At $t = 2$, $\frac{d^2 s}{dt^2} < 0$. At $t = -2$, $\frac{d^2 s}{dt^2} < 0$.
By (7.3), the stationary point $(0,0)$ is a minimum.
By (7.2), the stationary points $(2,16)$ and $(-2,16)$ are maximum.

By combining all the results we have the graph:



6. The function w crosses the vertical axis when $t = 0$:

$$w = 1 - \cos(50 \times 0) = 1 - \cos(0) = 1 - 1 = 0$$

Thus the graph goes through the origin $(0,0)$. To see where the graph meets the horizontal axis, we need to plot $w = 0$, so that $1 - \cos(50t) = 0$

$$\cos(50t) = 1, \quad 50t = \cos^{-1}(1) = 0,\ 2\pi$$

$$t = 0,\ \pi/25$$

The graph meets the t axis at the points (0,0) and $(0,\ \pi/25)$.

We need to differentiate w to find the stationary points.

$$w = 1 - \cos(50t)$$

$$\frac{dw}{dt} = 0 - \underbrace{\left[-50\sin(50t)\right]}_{\text{by (6.13)}} = 50\sin(50t)$$

Putting this to zero gives:

$$50\sin(50t) = 0$$

$$\sin(50t) = 0 \qquad (*)$$

What are we trying to find?

The value of t which satisfies (*). Taking inverse $\sin, \sin^{-1}$, of both sides:

$$50t = \sin^{-1}(0)$$

$$50t = 0,\ \pi,\ 2\pi$$

Remember $\sin$ is zero at all whole number multiplies of π. Dividing through by 50:

$$t = 0,\ \frac{\pi}{50},\ \frac{2\pi}{50} = \frac{\pi}{25}$$

Stationary points of w occurs at $t = 0,\ t = \dfrac{\pi}{50}$ and $\dfrac{\pi}{25}$. To decide their nature we differentiate again.

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left[50\sin(50t)\right] = 50^2\cos(50t) \qquad (\dagger)$$

Substituting the following t values into ($\dagger$):

At $t = 0,\ \dfrac{d^2w}{dt^2} = 50^2 > 0$ gives a local min (by (7.3)).

At $t = \dfrac{\pi}{50},\ \dfrac{d^2w}{dt^2} = 50^2\cos\left(50 \times \dfrac{\pi}{50}\right) = 50^2\cos(\pi) = -50^2 < 0$ gives a local max (by (7.2)).

(6.13) $\left[\cos(kt)\right]' = -k\sin(kt)$

(7.2) $w' = 0,\ w'' < 0$ maximum

(7.3) $w' = 0,\ w'' > 0$ minimum

At $t=\dfrac{\pi}{25}$, $\dfrac{d^2w}{dt^2}=50^2\cos\left(50\times\dfrac{\pi}{25}\right)=50^2\cos(2\pi)=50^2>0$ gives a local min (by (7.3)).

How do we find w for these values of t?

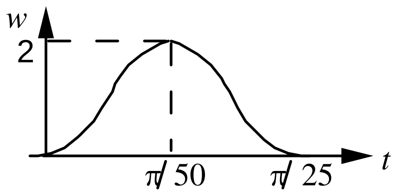
Substitute the t values into $w=1-\cos(50t)$:

At $t=0$, $w=1-\cos(0)=0$, min

At $t=\dfrac{\pi}{50}$, $w=1-\cos\left(50\times\dfrac{\pi}{50}\right)=1-\cos(\pi)=2$, max

At $t=\pi/25$, $w=0$, min

Combining the above we have:



7. First we find where the function, p, crosses the vertical axis, that is when $t=0$:

$p=10\sin^2(0)=0$

Thus the graph goes through the origin.

Next we find where the graph cuts the t axis, how?

Put $p=0$:

$$10\sin^2(t)=0,\ \ \sin^2(t)=0,\ \ \sin(t)=0$$

Taking the inverse $\sin$ of both sides: $t=0$, π and 2π

What conclusion can we draw from this?

The graph goes through the points $(0,0)$, $(\pi,0)$ and $(2\pi,0)$.

How do we find the maximum, minimum of $p=10\sin^2(t)$?

First we find the stationary points by putting $\dfrac{dp}{dt}=0$.

$$p=10\sin^2(t)$$

$$\frac{dp}{dt}=10\underbrace{\left[2\sin(t)\cos(t)\right]}_{\text{by (6.14)}}=10\left[\sin(2t)\right]=0$$

How do we extract t values out of $10\sin(2t)=0$?

Divide by 10 and take inverse $\sin$ of both sides:

$$\sin(2t)=0$$

$$2t=\sin^{-1}(0)$$

$$2t=0,\ \pi,\ 2\pi,\ 3\pi \text{ and } 4\pi$$

$$t=0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2} \text{ and } 2\pi \qquad \left[\text{Dividing by } 2\right]$$

Stationary points of $p=10\sin^2(t)$ occurs at $t=0$, $\dfrac{\pi}{2}$, π, $\dfrac{3\pi}{2}$ and 2π.

How do we find whether these points are maximum, minimum or points of inflexion?

Use the second derivative test:

$$\frac{dp}{dt}=10\sin(2t),\qquad \frac{d^2p}{dt^2}=20\cos(2t)$$

Substituting the t values:

$t = 0;\quad \dfrac{d^2p}{dt^2} = 20 > 0$ gives a local minimum (by (7.3))

$t = \dfrac{\pi}{2};\quad \dfrac{d^2p}{dt^2} = 20\cos\left(\dfrac{2\pi}{2}\right) = -20 < 0$ gives a local maximum (by(7.2))

$t = \pi;\quad \dfrac{d^2p}{dt^2} = 20\cos(2\pi) = 20 > 0$ gives a local minimum

$t = \dfrac{3\pi}{2};\quad \dfrac{d^2p}{dt^2} = 20\cos\left(2.\dfrac{3\pi}{2}\right) = -20 < 0$ gives a local maximum

$t = \pi;\quad \dfrac{d^2p}{dt^2} = 20\cos(2.2\pi) = 20 > 0$ gives a local minimum

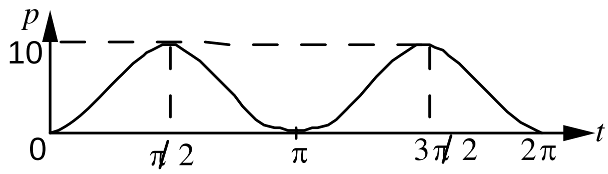
We have to evaluate p at these t values, how?

We have already established values of p for $t = 0,\ \pi$ and 2π because this is where the graph cuts the t-axis, that is $p = 0$. For the other 2 values of t we substitute into $p = 10\sin^2(t)$.

$t = \dfrac{\pi}{2},\qquad p = 10\left[\sin\left(\dfrac{\pi}{2}\right)\right]^2 = 10\qquad$ (max)

$t = \dfrac{3\pi}{2},\qquad p = 10\left[\sin\left(\dfrac{3\pi}{2}\right)\right]^2 = 10\qquad$ (max)

The points $(0,0),(\pi,0)$ and $(2\pi,0)$ are minimum points and $\left(\dfrac{\pi}{2},10\right),\left(\dfrac{3\pi}{2},10\right)$ are maximum points. Thus:



8. Similar to **EXAMPLE 7**. At $\left(\dfrac{\pi}{2}, 5\pi\right)$ and $\left(\dfrac{3\pi}{2}, 15\pi\right)$ we have horizontal points of inflexion of the form . Graph is very similar to Fig 15.

9. (i) As $t \to \infty, p \to 0$ because e^{-2t} term is predominate and is decaying.

(ii) We have $p = 0$ when $\left(t^2 - 2t\right)e^{-2t} = 0$. Remember the exponential function is never zero, so

$$t^2 - 2t = 0$$

$$t(t-2) = 0 \text{ gives } t = 0,\ t = 2$$

(iii) *We need to differentiate $p = \left(t^2 - 2t\right)e^{-2t}$, how?*

Need to use the product rule (6.31):

$$u = t^2 - 2t \qquad v = e^{-2t}$$

$$u' = 2t - 2 \qquad v' = -2e^{-2t}$$

(7.2) $\qquad p' = 0,\ p'' < 0$ maximum

(7.3) $\qquad p' = 0,\ p'' > 0$ minimum

Applying (6.31)

$$\frac{dp}{dt} = (2t-2)e^{-2t} - 2e^{-2t}\left(t^2 - 2t\right)$$

$$= \left[2t - 2 - 2\left(t^2 - 2t\right)\right]e^{-2t}$$

$$= \left[2t - 2 - 2t^2 + 4t\right]e^{-2t}$$

$$\frac{dp}{dt} = \left[-2 + 6t - 2t^2\right]e^{-2t}$$

For stationary points, $\frac{dp}{dt} = 0$:

$$\left(-2 + 6t - 2t^2\right)e^{-2t} = 0$$

$$-2 + 6t - 2t^2 = 0$$

Multiplying through by $-\frac{1}{2}$: $1 - 3t + t^2 = 0$

Thus we need to solve a quadratic equation: $t^2 - 3t + 1 = 0$

We use the quadratic formula with $a = 1$, $b = -3$ and $c = 1$.

$$t = \frac{3 \pm \sqrt{9 - (4\times 1\times 1)}}{2} = \frac{3 \pm \sqrt{5}}{2} = 0.382,\ \ 2.618 \quad \text{(3 d.p.)}$$

To check for maximum, minimum we apply the second derivative test on:

$$\frac{dp}{dt} = \left(-2 + 6t - 2t^2\right)e^{-2t}$$

Need to apply the product rule again, (6.31):

$$u = -2 + 6t - 2t^2 \qquad v = e^{-2t}$$

$$u' = 6 - 4t \qquad v' = -2e^{-2t}$$

$$\frac{d^2p}{dt^2} = (6-4t)e^{-2t} - 2e^{-2t}\left(-2 + 6t - 2t^2\right)$$

$$= \left[6 - 4t - 2\left(-2 + 6t - 2t^2\right)\right]e^{-2t}$$

$$\frac{d^2p}{dt^2} = \left[6 - 4t + 4 - 12t + 4t^2\right]e^{-2t} = \left[4t^2 - 16t + 10\right]e^{-2t} \qquad (*)$$

At $t = 0.382$,

$$\frac{d^2p}{dt^2} = \left[\left(4\times 0.382^2\right) - \left(16\times 0.382\right) + 10\right]e^{-(2\times 0.382)} > 0$$

By (7.3) at $t = 0.382$, p has a minimum value of

$$p = \left(0.382^2 - 2\times 0.382\right)e^{-(2\times 0.382)} = -0.288$$

(iv) For $t = 2.618$, substitute this value into (*),

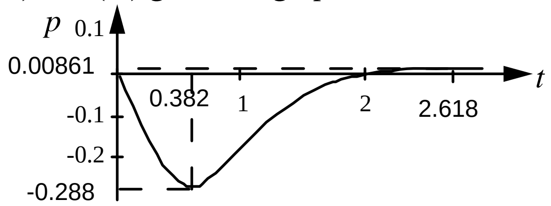
$$\frac{d^2p}{dt^2} = \left[\left(4\times 2.168^2\right) - \left(16\times 2.618\right) + 10\right]e^{-(2\times 2.618)} < 0$$

By (7.2), at $t = 2.618$, p has a maximum value of

$$p = \left(2.618^2 - 2\times 2.618\right)e^{-(2\times 2.618)} = 0.00861$$

(7.2) $\quad p' = 0,\ \ p'' < 0$ maximum

(v) Combining (ii), (iii) and (iv) gives the graph:



10. Similar to **EXAMPLE 5**. Graph of Fig 9 with max of e^{-1} at $t = 0.2$