Complete solutions to Exercise 7(b)

1. Since the perimeter =100 we have

$$2x+2y=100$$
$$x+y=50 \qquad \left[\text{Dividing both sides by 2}\right]$$
$$\vdots$$
$$y=50-x \qquad (*)$$

The area $A=xy$, substituting $y=50-x$ gives:

$$A=x\left(50-x\right)$$
$$A=50x-x^2$$

Differentiating to find the stationary point:

$$\frac{dA}{dx}=50-2x=0 \text{ gives } x=25m$$
$$\frac{d^2A}{dx^2}=-2<0$$

By (7.2), $x=25m$ gives maximum area. Substituting this value into (*):

$$y=50-25=25m$$

It seems as if the area constraint by the fence needs to be a square, $x=y=25m$.

2. From the fencing of $240m$ around the field we have

$$2x+y=240$$
$$y=240-2x \qquad (\dagger)$$

The area A is given by

$$A=xy$$
$$=x(240-2x) \qquad \text{by } (\dagger)$$
$$=240x-2x^2$$

For stationary points:

$$\frac{dA}{dx}=240-4x=0,\ \ 4x=240,\ \ x=60m$$

To show maximum we need to differentiate again

$$\frac{d^2A}{dx^2}=-4<0$$

By (7.2), when $x=60m$ we have maximum area. To find y we substitute $x=60$ into $(\dagger)$;

$$y=240-\left(2\times 60\right)=120m$$

Observe that y is twice the length of x, $x=60m$ and $y=120m$ gives maximum area.

3. We can rewrite $\dfrac{C_D+kC_L^2}{C_L}$ as

$$\frac{C_D}{C_L}+\frac{kC_L^2}{C_L}=C_DC_L^{-1}+kC_L$$

Differentiating this with respect to C_L gives:

$$\frac{d}{dC_L}\left(C_DC_L^{-1}+kC_L\right)=-C_DC_L^{-2}+k=k-C_DC_L^{-2}$$

(7.2) $A'=0,\ A''<0$ maximum

For stationary point this is equal to zero:

$$k - C_D C_L^{-2} = 0 \quad \text{gives} \quad k = C_D C_L^{-2} = \frac{C_D}{C_L^2}$$

Rearranging gives $C_L^2 = \dfrac{C_D}{k}$. Taking the square root of both sides:

$$C_L = \sqrt{\frac{C_D}{k}} = \left(\frac{C_D}{k}\right)^{1/2}$$

To check that this value of C_L gives a minimum we need to differentiate again:

$$\frac{d}{dC_L}\left(k - C_D C_L^{-2}\right) = -(-2)C_D C_L^{-3} = \frac{2C_D}{C_L^3} \qquad (\dagger)$$

Substituting $C_L = \left(\dfrac{C_D}{k}\right)^{1/2}$ into ($\dagger$) gives:

$$\frac{2C_D}{\left[\left(C_D/k\right)^{1/2}\right]^3} = \frac{2C_D}{\left(C_D/k\right)^{3/2}} = \frac{2C_D}{C_D{}^{3/2}/k^{3/2}} = \frac{2}{C_D{}^{1/2}/k^{3/2}} = \frac{2k^{3/2}}{C_D{}^{1/2}} = \frac{2\left(k^3\right)^{1/2}}{C_D{}^{1/2}} = 2\left(\frac{k^3}{C_D}\right)^{1/2}$$

$$2\left(\frac{k^3}{C_D}\right)^{1/2} = 2\sqrt{\frac{k^3}{C_D}} > 0 \quad \left(\text{taking the positive square root}\right)$$

Hence by (7.3) when $C_L = \sqrt{\dfrac{C_D}{k}}$ we have minimum drag.

4. We have

$$\pi r^2 h = 8$$
$$h = \frac{8}{\pi r^2} \qquad (\dagger)$$

The surface area, A, is given by the base, πr^2, and the curved area, $2\pi rh$, hence

$$A = \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r\left(\frac{8}{\pi r^2}\right) = \pi r^2 + \frac{16}{r}$$
$$A = \pi r^2 + 16r^{-1}$$

For stationary points:

$$\frac{dA}{dr} = 2\pi r - 16r^{-2} = 0$$
$$2\pi r = \frac{16}{r^2}$$
$$r^3 = \frac{16}{2\pi} = \frac{8}{\pi}$$

Take the cube root of both sides: $r = \sqrt[3]{\dfrac{8}{\pi}} = \dfrac{\sqrt[3]{8}}{\sqrt[3]{\pi}} = \dfrac{2}{\pi^{1/3}}$

To find whether this stationary point is a maximum or minimum we use the second derivative test:

$$\frac{dA}{dr} = 2\pi r - 16r^{-2} \qquad \frac{d^2A}{dr^2} = 2\pi + 32r^{-3}$$

(7.3) $A' = 0,\ A'' > 0$ minimum

Substituting $r=\frac{2}{\pi^{1/3}}$ into $\frac{d^2A}{dr^2}$ gives $\frac{d^2A}{dr^2}>0$. By (7.3), when $r=\frac{2}{\pi^{1/3}}\,m$ we have a minimum surface area. *How do we find h?*

Substitute $r=\frac{2}{\pi^{1/3}}$ into (†):

$$h=\frac{8}{\pi\left(2/\pi^{1/3}\right)^2}$$

$$=\frac{8}{\pi\left(2^2/\pi^{2/3}\right)}=\frac{2}{\pi^{1/3}}$$

The height and radius are equal.

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5. Similar to **EXAMPLE 9** with 1000 replaced by V. Volume $V=\pi r^2h$, gives

$$h=\frac{V}{\pi r^2} \qquad (*)$$

Surface Area $A=2\pi r^2+2\pi rh=2\pi r^2+2\pi r\left(\frac{V}{\pi r^2}\right)$

$$A=2\pi r^2+2Vr^{-1}$$

$$\frac{dA}{dr}=4\pi r-2Vr^{-2}=0$$

$$4\pi r=\frac{2V}{r^2}$$

$$r^3=\frac{2V}{4\pi}=\frac{V}{2\pi},\quad r=\sqrt[3]{\frac{V}{2\pi}}=\left(\frac{V}{2\pi}\right)^{1/3}$$

To show we have minimum surface area we have to differentiate again

$$\frac{d^2A}{dr^2}=4\pi+4Vr^{-3}$$

$$=4\pi+\frac{4V}{r^3}>0$$

$\frac{d^2A}{dr^2}$ is going to be positive because r is radius and is therefore positive.

By (7.3), when $r=\left(\frac{V}{2\pi}\right)^{1/3}$ we have minimum surface area.

To find h we substitute $r=\left(\frac{V}{2\pi}\right)^{1/3}$ into (*) and obtain $h=2r$.

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6. Factorizing the given equation gives:

$$M=\frac{W}{2}\left(Lx-x^2\right)$$

$$\frac{dM}{dx}=\frac{W}{2}\left(L-2x\right)=0,\text{ so } L-2x=0 \text{ which gives } x=\frac{L}{2}$$

$$\frac{d^2M}{dx^2}=\frac{W}{2}(-2)=-W<0$$

(7.2)	$M'=0,\ M''<0$	maximum
(7.3)	$A'=0,\ A''<0$	minimum

By (7.2), the bending moment is a maximum at $x=\frac{L}{2}$. The maximum value of M is evaluated by substituting $x=L/2$ into M:

$$\begin{aligned} M &= \frac{W}{2}\left(Lx-x^2\right) \\ &= \frac{W}{2}\left(L.\frac{L}{2}-\left(\frac{L}{2}\right)^2\right) \\ &= \frac{W}{2}\left[\frac{L^2}{2}-\frac{L^2}{4}\right]=\frac{W}{2}\left(\frac{L^2}{4}\right)=\frac{WL^2}{8} \end{aligned}$$

7. We have

$$y=\frac{W}{6EI}\left(3L^2x-x^3\right) \qquad (*)$$

For stationary points

$$\frac{dy}{dx}=\frac{W}{6EI}\left(3L^2-3x^2\right)=0,\ 3L^2-3x^2=0,\ x^2=L^2,\ x=\pm L$$

x cannot be $-L$ because L is a length. Hence $x=L$.
For maximum;

$$\frac{d^2y}{dx^2}=\frac{W}{6EI}\left(-6x\right)$$

$x=L$ $$\frac{d^2y}{dx^2}=-\frac{W}{EI}L<0$$

By (7.2), at $x=L$ we have maximum deflection.
Maximum deflection can be evaluated by substituting $x=L$ into (*)

$$\begin{aligned} y &= \frac{W}{6EI}\left(3L^2L-L^3\right) \\ &= \frac{W}{6EI}\left(3L^3-L^3\right) \\ &= \frac{W}{6EI}\left(2L^3\right)=\frac{WL^3}{3EI} \end{aligned}$$

8. Clearly by looking at Fig 20 it looks as if the maximum deflection will occur furthest from the fixed end, $x=L$. We need to show this by using differentiation.

$$y=\frac{W}{24EI}\left[x^4-4Lx^3+4L^2x^2\right]$$

$$\frac{dy}{dx}=\frac{W}{24EI}\left[4x^3-12Lx^2+8L^2x\right]\underbrace{=}_{\text{factorizing a factor of } 4x}\frac{4Wx}{24EI}\left[x^2-3Lx+2L^2\right]=0$$

$$\begin{aligned} x=0 \text{ or } x^2-3Lx+2L^2 &= 0 \\ (x-2L)(x-L) &= 0 \\ x=2L,\ x &= L \end{aligned}$$

We have a stationary points at $x=0, x=L$ and $x=2L$.
x cannot be $2L$ because the cantilever is only of length L.
Also at $x=0$, the fixed end, there is no deflection (minimum). We need to show that at $x=L$ we do have maximum deflection.

(7.2) $y'=0,\ y''<0$ maximum

$$\frac{dy}{dx}=\frac{W}{24EI}\left[4x^3-12Lx^2+8L^2x\right]$$

$$\frac{d^2y}{dx^2}=\frac{W}{24EI}\left[12x^2-24Lx+8L^2\right]$$

Substituting $x=L$ gives $\frac{d^2y}{dx^2}=\frac{W}{24EI}\left[12L^2-24L^2+8L^2\right]=\frac{W}{24EI}\left[-4L^2\right]<0$

By (7.2), at $x=L$ we have maximum deflection. The value of this deflection is

$$y=\frac{W}{24EI}\left[L^4-4L^4+4L^2L^2\right]$$

$$=\frac{W}{24EI}\left[L^4\right]=\frac{WL^4}{24EI}$$

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9. We have

$$p=Tv-mv^3$$

$$\frac{dp}{dv}=T-3mv^2=0,\ 3mv^2=T \text{ transposing gives } v=\sqrt{\frac{T}{3m}}$$

To show maximum for this value of v:

$$\frac{d^2p}{dv^2}=-6mv=-6m\sqrt{\frac{T}{3m}}<0$$

Hence by (7.2), $v=\sqrt{\frac{T}{3m}}$ gives maximum power.

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10. (i) $T=3\sin(2\theta)+6\sin(\theta)$

$$\frac{dT}{d\theta}=6\cos(2\theta)+6\cos(\theta)$$

$$=6\underbrace{\left(2\cos^2(\theta)-1\right)}_{\text{by (4.54)}}+6\cos(\theta)$$

$$=12\cos^2(\theta)+6\cos(\theta)-6$$

We need to solve $12\cos^2(\theta)+6\cos(\theta)-6=0$

Let $x=\cos(\theta)$:

$$12x^2+6x-6=0$$

$$2x^2+x-1=0 \qquad [\text{Dividing by } 6]$$

$$(2x-1)(x+1)=0 \text{ gives } x=1/2 \text{ or } x=-1$$

If $\cos(\theta)=1/2$ then $\theta=\pi/3$. If $\cos(\theta)=-1$ then θ lies outside the range $0\le\theta<\pi$. So θ can only be $\pi/3$.

$$\frac{d^2T}{d\theta^2}=-12\sin(2\theta)-6\sin(\theta)$$

At $\theta=\pi/3$, $\frac{d^2T}{d\theta^2}=-12\sin\left(\frac{2\pi}{3}\right)-6\sin\left(\frac{\pi}{3}\right)<0$

By (7.2), at $\theta=\pi/3$ we have maximum torque.

(4.54) $\cos(2x)=2\cos^2(x)-1$

(7.2) $y'=0,\ y''<0$ maximum

(ii) $T = 3\sin\left(\frac{2\pi}{3}\right) + 6\sin\left(\frac{\pi}{3}\right) = \frac{9\sqrt{3}}{2} \approx 7.8Nm$

11. (i) Similar to **EXAMPLE 10**.

$$P = i_1^2 R_1 + \left(i - i_1\right)^2 R_2$$

For stationary points:

$$\frac{dP}{di_1} = 2i_1 R_1 + 2\left(i - i_1\right) R_2 \left(-1\right) = 0$$

$$i_1 = \frac{iR_2}{R_1 + R_2} \qquad (*)$$

To establish this stationary point is a minimum, use the 2nd derivative:

$$\frac{d^2 P}{di_1^2} = 2R_1 + 2R_2 > 0 \qquad \left(\text{because } R_1 > 0 \text{ and } R_2 > 0\right)$$

By (7.3) (*) gives min P.

(ii) Substitute for i_1 from (*) into $iR = i_1 R_1$ gives the required result.