Complete solutions to Exercise 7(c)

1. Expanding out the brackets gives:

$$y = 8x^2 - x^4$$

$$\frac{dy}{dx} = 16x - 4x^3 = 4x\left(4 - x^2\right) \qquad \left[\text{Factorizing}\right]$$

For stationary points $4x\left(4 - x^2\right) = 0$. This gives $x = 0$ or

$$4 - x^2 = 0,\ x^2 = 4,\ x = \pm 2$$

When $x = 0$, $y = 0$

$$x = 2,\quad y = \left(8 \times 2^2\right) - 2^4 = 16$$

$$x = -2,\quad y = \left(8 \times (-2)^2\right) - (-2)^4 = 16$$

$(0,0)$, $(2,16)$ and $(-2,16)$ are stationary points of y. How do we distinguish which one of these is a local maximum or local minimum?

Use (7.7) or (7.8) for $x = 0$, $x = 2$ and $x = -2$:

$$\frac{dy}{dx} = 4x\left(4 - x^2\right)$$

For $x < 0$, $\frac{dy}{dx} = (-)(+) < 0$ and for $x > 0$, $\frac{dy}{dx} = (+)(+) > 0$.

Hence by (7.8) the stationary point $(0,0)$ is a local minimum.

For $x = 2$;

If $x < 2$, then $\frac{dy}{dx} = (+)(+) > 0$. If $x > 2$, then $\frac{dy}{dx} = (+)(-) < 0$.

By (7.7) the stationary point $(2,16)$ is a local maximum. For $x = -2$;

If $x < -2$ then $\frac{dy}{dx} = (-)(-) > 0$. If $x > -2$ then $\frac{dy}{dx} = (-)(+) < 0$.

By (7.7) the stationary point $(-2,16)$ is a local maximum. You can try particular values close to these stationary points.

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2. First we need to find the stationary points by differentiating and putting the result to zero.

We differentiate $a = \frac{10r + 1}{5r^2 + 3150}$ by the quotient rule, (6.32):

$$u = 10r + 1 \qquad v = 5r^2 + 3150$$

$$u' = 10 \qquad v' = 10r$$

$$\frac{da}{dr} = \frac{10\left(5r^2 + 3150\right) - (10r + 1)10r}{\left(5r^2 + 3150\right)^2}$$

$$= \frac{50r^2 + 31500 - 100r^2 - 10r}{\left(5r^2 + 3150\right)^2} = \frac{31500 - 10r - 50r^2}{\left(5r^2 + 3150\right)^2}$$

For $\frac{da}{dr} = 0$ we have $31500 - 10r - 50r^2 = 0$

Divide both sides by 10:

$$3150 - r - 5r^2 = 0$$

$$5r^2 + r - 3150 = 0 \quad [\text{Multiply by } -1]$$

(6.32) $\qquad (u/v)' = (u'v - uv')/v^2$

Solving this quadratic equation; substituting $a = 5,\ b = 1$ and $c = -3150$ into (1.16):

$$r = \frac{-1 \pm \sqrt{1^2 + (4 \times 5 \times 3150)}}{10} = 25 \text{ or } -25.2$$

The gear ratio $r = 25$.
Thus $r = 25$ gives a stationary point of a, but how do we know that this value of r gives maximum acceleration?
We can use the second derivative test but we have

$$\frac{da}{dr} = \frac{31500 - 10r - 50r^2}{\left(5r^2 + 3150\right)^2}$$

and differentiating this expression seems horrendous. Easier to use the first derivative test (7.7).
The denominator of $\frac{da}{dr}$ is positive, so we only need to examine the sign of the numerator for $r < 25$ and $r > 25$.
For $r < 25$, try $r = 24$:

$$31500 - (10 \times 24) - \left(50 \times 24^2\right) = 2460 > 0, \text{ so } \frac{da}{dr} > 0$$

For $r > 25$, try $r = 26$:

$$31500 - (10 \times 26) - \left(50 \times 26^2\right) = -2560 < 0, \text{ so } \frac{da}{dr} < 0$$

By (7.7) we have maximum acceleration at $r = 25$.

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3. We have

$$P = \frac{V^2 R_L}{\left(R + R_L\right)^2}$$

By (6.32)

$$\frac{dP}{dR_L} = \frac{V^2\left(R + R_L\right)^2 - 2V^2 R_L\left(R + R_L\right)}{\left(R + R_L\right)^4}$$

$$\underbrace{=}_{\text{factorizing top}} \frac{\left(R + R_L\right)\left[V^2\left(R + R_L\right) - 2V^2 R_L\right]}{\left(R + R_L\right)^4}$$

$$= \frac{V^2 R + V^2 R_L - 2V^2 R_L}{\left(R + R_L\right)^3} \qquad \left[\text{Cancelling } \left(R + R_L\right)\right]$$

$$= \frac{V^2 R - V^2 R_L}{\left(R + R_L\right)^3} \qquad \left[\text{Simplifying Numerator}\right]$$

$$\frac{dP}{dR_L} = \frac{V^2\left(R - R_L\right)}{\left(R + R_L\right)^3}$$

The numerator $V^2\left(R - R_L\right) = 0$ gives $\frac{dP}{dR_L} = 0$. Hence $R - R_L = 0$, since $V \neq 0$ otherwise we would have no power.

(1.16) $\qquad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(6.32) $\qquad (u/v)' = (u'v - uv')/v^2$

To show $R = R_L$ produces maximum power, we can use the first derivative test (7.7) because it is painless compared to differentiating

$$\frac{dP}{dR_L} = \frac{V^2\left(R - R_L\right)}{\left(R + R_L\right)^3}$$

Only need to inspect the sign of $R - R_L$ because the remaining terms are positive.

If $R_L < R$ then $R - R_L > 0$ so $\dfrac{dP}{dR_L} > 0$.

If $R_L > R$ then $R - R_L < 0$ so $\dfrac{dP}{dR_L} < 0$.

By (7.7), $R = R_L$ gives maximum power transfer.

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4. We have

$$E = \frac{Vb}{ba - a^2} = Vb\left(ba - a^2\right)^{-1}$$

Differentiating

$$\frac{dE}{da} = -Vb\left(ba - a^2\right)^{-2}\left(b - 2a\right) = \frac{-Vb\left(b - 2a\right)}{\left(ba - a^2\right)^2}$$

$$\frac{dE}{da} = \frac{Vb\left(2a - b\right)}{\left(ba - a^2\right)^2} \qquad \left(\begin{array}{l}\text{taking out a negative sign}\\ \text{from } \left(2a - b\right)\end{array}\right)$$

For stationary point, $\dfrac{dE}{da} = 0$ so the numerator $= 0$:

$$Vb\left(2a - b\right) = 0$$
$$2a - b = 0$$
$$a = \frac{b}{2}$$

How can we show that $a = \dfrac{b}{2}$ produces minimum E?

Use the first derivative test (7.8):

$$\frac{dE}{da} = \frac{Vb\left(2a - b\right)}{\left(ba - a^2\right)^2}$$

We only need to check the sign of $2a - b$ because the remaining terms are positive.

If $a < \dfrac{b}{2}$ then $2a - b < 0$, so $\dfrac{dE}{da} < 0$.

If $a > \dfrac{b}{2}$ then $2a - b > 0$, so $\dfrac{dE}{da} > 0$.

By (7.8), $a = \dfrac{b}{2}$ gives the minimum electric stress.

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5. The maximum value is $E = 2\pi fk$ because $\cos$ function lies between -1 and +1 $\left(-1 \le \cos\left(2\pi ft\right) \le 1\right)$.