Complete solutions to Exercise 7(g)

1. (a) The first 4 terms are determined by putting $a=1$, $b=x$ and $n=20$ into (7.23):

$$(1+x)^{20}=1^{20}+\left(20\times 1^{19}\times x\right)+\left[\frac{20(19)}{2!}\right]1^{18}x^2+\left[\frac{20(19)(18)}{3!}\right]1^{17}x^3+\dots$$

$$=1+20x+190x^2+1140x^3+\dots$$

(b) Putting $a=x$, $b=y$ and $n=15$ into (7.23) gives:

$$(x+y)^{15}=x^{15}+\left(15\times x^{14}\times y\right)+\left[\frac{15(14)}{2!}\right]x^{13}y^2+\left[\frac{15(14)(13)}{3!}\right]x^{12}y^3$$

$$=x^{15}+15x^{14}y+105x^{13}y^2+455x^{12}y^3+\dots$$

(c) Putting $a=2$, $b=-Z$ and $n=10$ into (7.23) gives:

$$\left(2-Z\right)^{10}=2^{10}+\left[10\times 2^9\times(-Z)\right]+\left[\frac{10(9)}{2!}\right]2^8(-Z)^2+\left[\frac{10(9)(8)}{3!}\right]2^7(-Z)^3+\dots$$

$$=1024+\left[10\times 2^9\times(-Z)\right]+\left[45\times 2^8\right]\left(Z^2\right)+\left[120\times 2^7\right]\left(-Z^3\right)+\dots$$

$$=1024-5120Z+11520Z^2-15360Z^3+\dots$$

2. Rewrite $\dfrac{1}{1-Z}=(1-Z)^{-1}$. Substituting $x=-Z$ and $n=-1$ into (7.24) gives:

$$(1-Z)^{-1}=1+[(-1)(-Z)]+\left[\frac{(-1)(-2)}{2!}(-Z)^2\right]$$

$$+\left[\frac{(-1)(-2)(-3)}{3!}(-Z)^3\right]+\left[\frac{(-1)(-2)(-3)(-4)}{4!}(-Z)^4\right]+\dots$$

$$=1+Z+Z^2+Z^3+Z^4+\dots$$

3. Similar to **EXAMPLE 25**. Putting $n=\dfrac{1}{2}$ and replacing x with $\left(\dfrac{Wx}{T}\right)^2$ into (7.24) gives:

$$\left[1+\left(\frac{Wx}{T}\right)^2\right]^{\frac{1}{2}}=1+\frac{1}{2}\left(\frac{Wx}{T}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{Wx}{T}\right)^4+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{Wx}{T}\right)^6\dots$$

$$=1+\frac{W^2x^2}{2T^2}-\frac{W^4x^4}{8T^4}+\frac{W^6x^6}{16T^6}\dots \text{ where } -1<\left(\frac{Wx}{T}\right)^2<1$$

4. (i) By applying the binomial theorem and writing the index as ½ we have

$\sqrt{1-x}=\left(1-x\right)^{1/2}$

$$=1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-x)^3+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(-x)^4+\cdots$$

(7.23) $$\left(a+b\right)^n=a^n+na^{n-1}b+\left[\frac{n(n-1)}{2!}\right]a^{n-2}b^2+\left[\frac{n(n-1)(n-2)}{3!}\right]a^{n-3}b^3+\dots$$

$$=1-\frac{1}{2}x-\frac{1}{8}(-x)^2+\frac{3}{48}(-x)^3+\left(\frac{-15}{384}\right)(-x)^4+\cdots \qquad [\text{Evaluating fractions}]$$

$$=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\cdots \qquad [\text{Simplifying fractions}]$$

(ii) We have $\sqrt{7}=\sqrt{9\left(1-\frac{2}{9}\right)}=3\sqrt{\left(1-\frac{2}{9}\right)}$. By using part (i) with $x=\frac{2}{9}$ we have

$$\sqrt{1-\frac{2}{9}}=\left(1-\frac{2}{9}\right)^{1/2}$$

$$=1-\frac{1}{2}\left(\frac{2}{9}\right)-\frac{1}{8}\left(\frac{2}{9}\right)^2-\frac{1}{16}\left(\frac{2}{9}\right)^3-\frac{5}{128}\left(\frac{2}{9}\right)^4-\cdots$$

$$=0.8819$$

Hence $\sqrt{7}=3\sqrt{\left(1-\frac{2}{9}\right)}=3\times 0.8819=2.6458$.

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5. We have $T=\frac{W\ell}{2}\left[1+\left(\frac{\ell}{4s}\right)^2\right]^{\frac{1}{2}}$. (Remember $\sqrt{\ }$ is to the index $\frac{1}{2}$). Let $x=\frac{\ell}{4s}$ and we first determine $\left(1+x^2\right)^{\frac{1}{2}}$. Using (7.24):

$$\left(1+x^2\right)^{\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(x^2)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^2)^3+\ldots$$

$$=1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\ldots \qquad (*)$$

Putting $x=\frac{\ell}{4s}$ into (*) gives:

$$\left[1+\left(\frac{\ell}{4s}\right)^2\right]^{\frac{1}{2}}=1+\frac{1}{2}\left(\frac{\ell}{4s}\right)^2-\frac{1}{8}\left(\frac{\ell}{4s}\right)^4+\frac{1}{16}\left(\frac{\ell}{4s}\right)^6\ldots$$

$$=1+\frac{\ell^2}{\left(2\times 4^2\right)s^2}-\frac{\ell^4}{\left(8\times 4^4\right)s^4}+\frac{\ell^6}{\left(16\times 4^6\right)s^6}\ldots$$

$$=1+\frac{\ell^2}{32s^2}-\frac{\ell^4}{2048s^4}+\frac{\ell^6}{65536s^6}\ldots \qquad [\text{Simplifying}]$$

Replacing $\left[1+\left(\frac{\ell}{4s}\right)^2\right]^{\frac{1}{2}}$ with the Right Hand Side into $T=\frac{W\ell}{2}\left[1+\left(\frac{\ell}{4s}\right)^2\right]^{\frac{1}{2}}$ gives:

(7.24) $$\left(1+x\right)^n=1+nx+\left[\frac{n(n-1)}{2!}\right]x^2+\left[\frac{n(n-1)(n-2)}{3!}\right]x^3+\ldots$$

$$T=\frac{W\ell}{2}\left[1+\frac{\ell^2}{32s^2}-\frac{\ell^4}{2048s^4}+\frac{\ell^6}{65536s^6}\ldots\right]$$

$$=\frac{W\ell}{2}+\frac{W\ell^3}{64s^2}-\frac{W\ell^5}{4096s^4}+\frac{W\ell^7}{131072s^6}\ldots$$

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6. Putting $x=\frac{1}{2}(\gamma-1)M^2$ and $n=\frac{\gamma}{\gamma-1}$ into (7.24) gives:

$$P=\left(1+\frac{1}{2}(\gamma-1)M^2\right)^{\frac{\gamma}{\gamma-1}}=\ 1+\frac{\gamma}{\gamma-1}\cdot\frac{1}{2}(\gamma-1)M^2+\left[\frac{1}{2!}\frac{\gamma}{\gamma-1}\cdot\left(\frac{\gamma}{\gamma-1}-1\right)\left(\frac{1}{2}(\gamma-1)M^2\right)^2\right]$$

$$+\left[\frac{1}{3!}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{\gamma}{\gamma-1}-1\right)\left(\frac{\gamma}{\gamma-1}-2\right)\left(\frac{1}{2}(\gamma-1)M^2\right)^3\right]+\ldots$$

$$=1+\frac{\gamma M^2}{2}+\left[\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{1}{\gamma-1}\right)\cdot\frac{1}{4}(\gamma-1)^2M^4\right]+\left[\frac{1}{6}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{1}{\gamma-1}\right)\left(\frac{2-\gamma}{\gamma-1}\right)\cdot\frac{1}{8}(\gamma-1)^3M^6\right]+\ldots$$

$$=1+\frac{\gamma M^2}{2}+\left[\frac{1}{8}\frac{\gamma}{(\gamma-1)^2}(\gamma-1)^2M^4\right]+\left[\frac{1}{48}\frac{\gamma(2-\gamma)}{(\gamma-1)^3}(\gamma-1)^3M^6\right]+\ldots$$

$$=1+\frac{\gamma M^2}{2}+\frac{\gamma M^4}{8}+\frac{\gamma(2-\gamma)}{48}M^6+\ldots$$ [Cancelling Common Terms]

$$P=1+\gamma\left(\frac{M^2}{2}+\frac{M^4}{8}+\frac{(2-\gamma)}{48}M^6+\ldots\right)$$ [Factorizing]

(7.24) $$\left(1+x\right)^n=1+nx+\left[\frac{n(n-1)}{2!}\right]x^2+\left[\frac{n(n-1)(n-2)}{3!}\right]x^3+\ldots$$