Complete solutions to Exercise 8(c)

1. (a) $\int \sin(7x+1)dx = -\frac{\cos(7x+1)}{7} + C$ [by (8.39)]

(b) $\int \cos(7x+1)dx = \frac{\sin(7x+1)}{7} + C$ [by (8.38)]

2. Using (8.38):

(a) $\int \cos(\omega t)\,dt = \frac{\sin(\omega t)}{\omega} + C$

(b) $\int \cos(\omega t + \theta)dt = \frac{\sin(\omega t + \theta)}{\omega} + C$

3. (a) $\int \sin(\omega t)\,dt \underbrace{=}_{\text{by (8.39)}} -\frac{\cos(\omega t)}{\omega} + C$

(b) $\int \sin(\omega t)\,d(\omega t) = -\cos(\omega t) + C$ (by (8.7) with $u = \omega t$)

4. (a) Differentiating $x^2 - 1$ with respect to x gives $2x$. Hence by using (8.42) we have

$$\int \frac{2x}{x^2-1}dx = \ln\left|x^2-1\right| + C$$

(b) Differentiating $x^3 - 3x^2 + 1$ with respect to x gives $3x^2 - 6x$. Using (8.42)

$$\int \frac{3x^2-6x}{x^3-3x^2+1}dx = \ln\left|x^3-3x^2+1\right| + C$$

5. We have $\cot(x) = \frac{\cos(x)}{\sin(x)}$. Differentiating $\sin(x)$ gives $\cos(x)$, hence

$$\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}dx$$

$$\underbrace{=}_{\text{by (8.42)}} \ln\left|\sin(x)\right| + C$$

6. (a) We have $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$. Differentiating $\cosh(x)$ gives $\sinh(x)$, so we can use (8.42):

$$\int \tanh(x)\,dx = \int \frac{\sinh(x)}{\cosh(x)}dx \underbrace{=}_{\text{by (8.42)}} \ln\left|\cosh(x)\right| + C$$

(b) We have $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$ and differentiating $\sinh(x)$ gives $\cosh(x)$. So

$$\int \coth(x)\,dx = \int \frac{\cosh(x)}{\sinh(x)}dx \underbrace{=}_{\text{by (8.42)}} \ln\left|\sinh(x)\right| + C$$

(8.7) $\int \sin(u)du = -\cos(u)$

(8.38) $\int \cos(kx+m)dx = \sin(kx+m)/k$

(8.39) $\int \sin(kx+m)dx = -\cos(kx+m)/k$

(8.42) $\int f'(x)/f(x) = \ln\left|f(x)\right|$

7. (a) Differentiating $7t - 1$ gives 7, so

$$\int \frac{dt}{7t-1} = \frac{1}{7}\int \frac{7dt}{7t-1}$$
$$= \frac{1}{7}\underbrace{\ln|7t-1|}_{\text{by (8.42)}} + C$$

(b) Differentiating $t^4 - 1$ with respect to t gives $4t^3$. We can write t^3 on the numerator as $\frac{1}{4}(4t^3)$. Hence

$$\int \frac{t^3}{t^4-1}dt = \frac{1}{4}\int\left(\frac{4t^3}{t^4-1}\right)dt$$
$$= \frac{1}{4}\underbrace{\ln|t^4-1|}_{\text{by (8.42)}} + C$$

(c) Differentiating $5-t^3$ with respect to t gives $-3t^2$. We can write t^2 as $-\frac{1}{3}(-3t^2)$

Thus

$$\int \frac{t^2}{5-t^3}dt = -\frac{1}{3}\int \frac{-3t^2}{5-t^3}dt$$
$$= -\frac{1}{3}\underbrace{\ln|5-t^3|+C}_{\text{by (8.42)}}$$

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8. We use $\int e^{kx+m}dx = e^{kx+m}/k + C$ in each case.

(a) $\int e^{11x+5}dx = \dfrac{e^{11x+5}}{11} + C$ (b) $\int e^{-2x+1000}dx = \dfrac{e^{-2x+1000}}{-2} + C = -\dfrac{e^{-2x+1000}}{2} + C$

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9. We have

$$v = \int(-g)dt = -gt + C \qquad (\dagger)$$

Substituting $t = 0,\ v = v_0$ gives

$$v_0 = -(g\times 0) + C = 0 + C,\ v_0 = C$$

Substituting $C = v_0$ into $(\dagger)$:

$$v = v_0 - gt$$

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10. Same as solution 9 with $v_0 = u$.

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11. Taking out the 10 gives:

$$s = 10\int(30t+1)^{-1/2}dt \qquad (*)$$

How do we integrate $(30t+1)^{-1/2}$?

Use substitution. Let $u = 30t+1$, remember we also need to replace the dt, how?

Differentiating:

$$u = 30t+1,\ \frac{du}{dt} = 30 \text{ gives } dt = \frac{du}{30}$$

Putting $u = 30t+1$ and $dt = \dfrac{du}{30}$ into (*) gives:

(8.42) $$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$$

$$s = 10\int u^{-1/2}\frac{du}{30}$$
$$= \frac{10}{30}\int u^{-1/2}du$$
$$= \frac{1}{3}\underbrace{\left(\frac{u^{-1/2+1}}{-1/2+1}\right)}_{\text{by (8.1)}} + C$$
$$= \frac{1}{3}\left(\frac{u^{1/2}}{1/2}\right) + C$$
$$= \frac{1}{3}\left(2u^{1/2}\right) + C$$
$$= \frac{2}{3}u^{1/2} + C$$
$$s = \frac{2}{3}\left(30t+1\right)^{1/2} + C \quad \text{(Replacing } u\text{)}$$

Using $t = 0,\ \ s = 2/3$ gives:

$$2/3 = 2/3\left[(30\times 0)+1\right]^{1/2} + C$$
$$2/3 = 2/3 + C \text{ gives } C = 0$$

Hence $s = \frac{2}{3}(30t+1)^{1/2}$.

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12. (i)

$$v = -6\int t\,dt = -6\left(\frac{t^2}{2}\right) + C$$
$$v = -3t^2 + C$$

Substituting $t = 0,\ \ v = 48$

$$48 = 0 + C \quad \text{gives} \quad C = 48$$

Hence $v = 48 - 3t^2$.

(ii) We need to find t for $v = 0$.

$$48 - 3t^2 = 0,\ \ 3t^2 = 48 \text{ which gives} \quad t = \sqrt{16} = 4 \text{ sec}$$

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13. Rearranging $k = \frac{P}{\rho^\gamma}$ we have $\rho^\gamma = \frac{P}{k}$. How can we find ρ on its own?

Taking γ - root of both sides:

$$\rho = \left(\frac{P}{k}\right)^{1/\gamma} = \frac{P^{1/\gamma}}{k^{1/\gamma}} = \frac{P^{1/\gamma}}{C} \text{ where } C = k^{1/\gamma}$$

Warning: This C is **not** the constant of integration. Putting $\rho = \frac{P^{1/\gamma}}{C}$ gives:

(8.1) $\int u^n du = \frac{u^{n+1}}{n+1}$

$$\begin{aligned}\int \frac{dP}{\rho} &= \int \frac{dP}{\left(P^{1/\gamma}/C\right)} \\ &= \int \frac{CdP}{P^{1/\gamma}} \\ &= C\int P^{-1/\gamma}dP \\ &= C\underbrace{\left(\frac{P^{-1/\gamma+1}}{-1/\gamma+1}\right)}_{\text{by (8.1)}}+D \qquad (\dagger)\end{aligned}$$

We use D as the constant of integration. Of course we can use any letter to represent the constant of integration. Simplifying $-1/\gamma+1$:

$$\begin{aligned}-\frac{1}{\gamma}+1 &= 1-\frac{1}{\gamma} \\ &= \underbrace{\frac{\gamma}{\gamma}}_{=1}-\frac{1}{\gamma} \\ &= \frac{\gamma-1}{\gamma}\end{aligned}$$

Replacing $-\frac{1}{\gamma}+1$ with $\frac{\gamma-1}{\gamma}$ in $(\dagger)$ gives:

$$\begin{aligned}\int \frac{dP}{\rho} &= C\left(\frac{P^{\frac{\gamma-1}{\gamma}}}{\frac{\gamma-1}{\gamma}}\right)+D \\ &= \frac{C\gamma P^{\frac{\gamma-1}{\gamma}}}{\gamma-1}+D \\ &= \frac{k^{1/\gamma}\gamma P^{\frac{\gamma-1}{\gamma}}}{\gamma-1}+D\end{aligned}$$

(8.1) $\int u^n du = \frac{u^{n+1}}{n+1}$