Complete solutions to Exercise 7(e)

1. (a) The gradient, m, of the tangent is evaluated by differentiating $y = x^2 - 3$:

$$\frac{dy}{dx} = 2x$$

At $x = 2$, $\frac{dy}{dx} = 2 \times 2 = 4$

So $m = 4$. Equation of tangent is of the form $y = 4x + c$. At $x = 2,\ y = 2^2 - 3 = 1$ so the tangent goes through the point $(2,1)$, that is when $x = 2,\ y = 1$.

$$1 = (4 \times 2) + c \text{ gives } c = -7$$

Equation of tangent is $y = 4x - 7$.

By (7.13), the gradient of normal $= -\frac{1}{4}$.

Equation of normal is of the form, $y = -\frac{1}{4}x + c_1$, and it also goes through the point $(2,1)$, that is $x = 2$ and $y = 1$. We have

$$1 = \left(-\frac{1}{4} \times 2\right) + c_1 \text{ gives } c_1 = \frac{3}{2}$$

Equation of normal is $y = -\frac{1}{4}x + \frac{3}{2} = \frac{6}{4} - \frac{1}{4}x = \frac{1}{4}(6 - x)$ [Factorizing]

(b) Differentiating $y = \cos(x)$ gives:

$$\frac{dy}{dx} = -\sin(x)$$

At $x = \frac{\pi}{2}$, $\frac{dy}{dx} = -\sin\left(\frac{\pi}{2}\right) = -1$. The gradient, $m = -1$, so the equation of the tangent has the form

$$y = -x + c$$

How can we find c?

The tangent goes through the point $x = \frac{\pi}{2}$ and $y = \cos\left(\frac{\pi}{2}\right) = 0$. Substituting these into $y = -x + c$ yields:

$$0 = -\frac{\pi}{2} + c \text{ gives } c = \frac{\pi}{2}$$

Hence the equation of the tangent is

$$y = -x + \frac{\pi}{2} = \frac{\pi}{2} - x$$

How do we find the equation of the normal?

By (7.13) the gradient of the normal $= \frac{-1}{-1} = 1$. Equation of the normal has the form

$$y = x + c_1$$

and also goes through the point $\left(\frac{\pi}{2}, 0\right)$ that is $x = \frac{\pi}{2}$, $y = 0$.

(7.13) Gradient of normal $= -\frac{1}{m}$

$$0 = \frac{\pi}{2} + c_1 \text{ gives } \quad c_1 = -\frac{\pi}{2}$$

Equation of normal is $y = x - \frac{\pi}{2}$.

(c) Differentiating $y = e^x$ gives $\frac{dy}{dx} = e^x$.

Substituting $x = 1$, $\frac{dy}{dx} = e^1 = e$

The gradient of the tangent $= e$ and is of the form

$$y = ex + c \qquad (*)$$

Substituting $x = 1$ into $y = e^x$ gives $y = e^1 = e$, so the tangent goes through the point $x = 1,\ y = e$ or $(1, e)$. Putting these values into (*)

$$e = e(1) + c \text{ gives } c = 0$$

Therefore the equation of the tangent is $y = ex$.

What is the gradient of the normal?

By (7.13):

$$\text{gradient } = -\frac{1}{e} = -e^{-1}$$

Equation of normal is of the form

$$y = -e^{-1}x + c_1 \qquad (**)$$

and it also goes through the point $x = 1$, $y = e$.

$$e = -e^{-1}(1) + c_1$$

Making c_1 the subject

$$c_1 = e + e^{-1}$$

Substituting this into (**) gives the equation of the normal:

$$y = -e^{-1}x + \left(e + e^{-1}\right)$$

$$= e + e^{-1}(1 - x)$$

(d) Differentiating $y = \ln(x)$ gives

$$\frac{dy}{dx} = \frac{1}{x}$$

At $x = 1$, $\frac{dy}{dx} = \frac{1}{1} = 1$

The gradient of tangent =1 and is of the form $y = x + c$.

At $x = 1,\ y = \ln(1) = 0$. The tangent goes through $(1, 0)$, hence

$$0 = 1 + c \text{ gives } c = -1$$

Equation of tangent is $y = x - 1$.

Gradient of normal $= -1$. Hence $y = -x + c_1$. The normal goes through the point $(1, 0)$, so we have

$$0 = -1 + c_1 \quad \text{therefore} \quad c_1 = 1$$

Equation of normal is $y = -x + 1 = 1 - x$.

(7.13) Gradient of normal $= -\frac{1}{m}$

2. Since the tangent is a straight line relationship between i and v we have

$$i = mv + c \qquad (*)$$

where m is the gradient and c is the i intercept. The gradient at $v = 1.6$ is given by differentiating $i = v^3$:

$$m = \frac{di}{dv} = 3v^2$$

Substituting $v = 1.6$ gives

$$m = 3 \times 1.6^2 = 7.68$$

Thus putting this into (*) gives:

$$i = 7.68v + c \qquad (**)$$

How can we find c ?

We know at $v = 1.6$, $i = 1.6^3 = 4.096$ (substituting v into the original equation).
Using (**)

$$4.096 = (7.68 \times 1.6) + c \text{ gives } c = -8.19$$

Hence the equation of the tangent is

$$i = 7.68v - 8.19$$

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3. Expanding the brackets:

$$i = 2 - 2e^{-2000t}$$

An expression for the gradient of the tangent is given by

$$\frac{di}{dt} = 0 - \underbrace{\left[2 \times (-2000)\right] e^{-2000t}}_{\text{by (6.11)}}$$

$$= 4000e^{-2000t}$$

At $t = 1 \times 10^{-3}$,

$$\frac{di}{dt} = 4000 \times e^{-2000 \times \left(1 \times 10^{-3}\right)} = 541.34 \qquad (= \text{gradient})$$

The equation of tangent is of the form

$$i = 541.34t + c \qquad (*)$$

We need to find c. The tangent goes through a value of $t = 1 \times 10^{-3}$, so

$$i = 2 - 2e^{-\left(2000 \times 1 \times 10^{-3}\right)} = 1.73$$

The tangent goes through the point $t = 1 \times 10^{-3}$ and $i = 1.73$.
Substituting these into (*):

$$1.73 = \left(541.34 \times 1 \times 10^{-3}\right) + c \text{ gives } c = 1.19$$

Hence

$$i = 541.34t + 1.19$$

(6.11) $$\left(e^{kx}\right)' = ke^{kx}$$