Complete solutions to Exercise 7(f)

1. Let $f(x)=\cos(x)$ then differentiating this several times and substituting $x=0$ gives:

$$f(x)=\cos(x) \qquad f(0)=\cos(0)=1$$
$$f'(x)=-\sin(x) \qquad f'(0)=-\sin(0)=0$$
$$f''(x)=-\cos(x) \qquad f''(0)=-\cos(0)=-1$$
$$f'''(x)=\sin(x) \qquad f'''(0)=\sin(0)=0$$
$$f^{(4)}(x)=\cos(x) \qquad f^{(4)}(0)=\cos(0)=1$$

Substituting these into (7.14) gives:

$$\cos(x)=1+0(x)+\frac{(-1)}{2!}x^2+\frac{0}{3!}x^3+\frac{1}{4!}x^4+\dots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$$

2. Let $f(x)=\ln(1+x)$, differentiating gives:

$$f(x)=\ln(1+x) \qquad f(0)=\ln(1+0)=0$$
$$f'(x)\underbrace{=}_{\text{by (6.18)}}\frac{1}{1+x}=(1+x)^{-1} \qquad f'(0)=(1+0)^{-1}=1$$
$$f''(x)=-(1+x)^{-2} \qquad f''(0)=-(1+0)^{-2}=-1$$
$$f'''(x)=2(1+x)^{-3} \qquad f'''(0)=2(1+0)^{-3}=2$$
$$f^{(4)}(x)=-6(1+x)^{-4} \qquad f^{(4)}(0)=-6(1+0)^{-4}=-6$$

Substituting these into (7.14):

$$\ln(1+x)=0+(1)x+\frac{(-1)}{2!}x^2+\frac{2}{3!}x^3+\frac{(-6)}{4!}x^4+\dots$$
$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \qquad -1<x\le 1$$

3. Our function is $f(x)=(1+x)^n$ so differentiating this gives:

$$f(x)=(1+x)^n \qquad f(0)=1$$
$$f'(x)=n(1+x)^{n-1} \qquad f'(0)=n$$
$$f''(x)=n(n-1)(1+x)^{n-2} \qquad f''(0)=n(n-1)$$
$$f'''(x)=n(n-1)(n-2)(1+x)^{n-3} \qquad f'''(0)=n(n-1)(n-2)$$

Substituting the above into (7.14) gives:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots \quad -1<x<1$$

Note that if n is a positive integer then the series terminates after $n+1$ terms and is valid for all x.

(7.14) $$f(x)=f(0)+xf'(0)+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\dots$$

4. By (4.67) we have

$$\cos^2(x)=\frac{1}{2}\left[\cos(2x)+1\right] \qquad (\dagger)$$

The series expansion of $\cos(2x)$ can be obtained by using (7.17) and replacing the x with $2x$:

$$\cos(2x)=1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots$$
$$=1-\frac{2^2x^2}{2!}+\frac{2^4x^4}{4!}-\frac{2^6x^6}{6!}+\dots$$
$$=1-\frac{4x^2}{2!}+\frac{16x^4}{4!}-\frac{64x^6}{6!}$$
$$\cos(2x)=1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}$$

You can find the simplified fraction by using your calculator. Substituting the last line into (†) gives:

$$\cos^2 x=\frac{1}{2}\left(1-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\dots+1\right)$$
$$=\frac{1}{2}\left(2-2x^2+\frac{2x^4}{3}-\frac{4x^6}{45}+\dots\right)$$
$$\cos^2 x=1-x^2+\frac{x^4}{3}-\frac{2x^6}{45}+\dots \qquad (\dagger\dagger)$$

How can we find the expansion of $\sin^2(x)$*?*

By (4.64) we have $\sin^2(x)=1-\cos^2(x)$, substituting the Right Hand Side of $(\dagger\dagger)$ for $\cos^2(x)$ into this gives:

$$\sin^2(x)=1-\left(1-x^2+\frac{x^4}{3}-\frac{2x^6}{45}+\dots\right)=(1-1)+x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\dots$$
$$\sin^2(x)=x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\dots$$

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5. (a) Substitute the Maclaurin series expansion for $\ln(1+x)$ by using (7.21)

$$\frac{\ln(1+x)}{x}=\frac{x-\frac{x^2}{2}+\frac{x^3}{3}\dots}{x}$$
$$=\frac{x\left(1-\frac{x}{2}+\frac{x^2}{3}\dots\right)}{x}=1-\frac{x}{2}+\frac{x^2}{3}\dots \qquad (\text{Cancelling } x's)$$
$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\left(1-\frac{x}{2}+\frac{x^2}{3}\dots\right)=1$$

(4.64) $\cos^2(x)+\sin^2(x)=1$

(7.21) $\ln(1+x)=x-x^2/2+x^3/3-\dots$

(b) Applying the Right Hand Side of (7.17) for $\cos(x)$ gives:

$$\frac{1-\cos(x)}{x^2}=\frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}...\right)}{x^2}$$

$$=\frac{\frac{x^2}{2!}-\frac{x^4}{4!}...}{x^2}$$

$$=\frac{x^2\left(\frac{1}{2!}-\frac{x^2}{4!}...\right)}{x^2}$$

$$=\frac{1}{2!}-\frac{x^2}{4!}...$$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{1}{2!}-\frac{x^2}{4!}...\right)$$

$$=\frac{1}{2!}=\frac{1}{2}$$

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6. (a) We need to find the Taylor series for $\ln(x)$ about the point $x=1$. Very similar to Example 21:

$f(x)=\ln(x)$	$f(1)=\ln(1)=0$
$f'(x)=\frac{1}{x}=x^{-1}$	$f'(1)=\frac{1}{1}=1$
$f''(x)=-x^{-2}=-\frac{1}{x^2}$	$f''(1)=-\frac{1}{1^2}=-1$
$f'''(x)=2x^{-3}=\frac{2}{x^3}$	$f'''(1)=\frac{2}{1^3}=2$
$f^{(4)}(x)=-6x^{-4}=-\frac{6}{x^4}$	$f^{(4)}(1)=-\frac{6}{1^4}=-6$

Substituting these values into (7.22) we have Taylor series for $\ln(x)$ at $x=1$:

$$f(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(2)}{3!}(x-1)^3+...$$

$$=0+(1)(x-1)+\frac{(-1)}{2!}(x-1)^2+\frac{(2)}{3!}(x-1)^3+\frac{(-6)}{4!}(x-1)^4+...$$

$$=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+... \qquad [\text{Simplifying}]$$

(b) The Taylor series for $f(x)=\frac{1}{x}$ at $x=1$ can be determined by differentiating $f(x)$ and then substituting the value $x=1$:

(7.17) $\cos(x)=1-x^2/2!+x^4/4!...$

$$f(x)=\frac{1}{x}=x^{-1} \qquad f(1)=\frac{1}{1}=1$$

$$f'(x)=-x^{-2}=-\frac{1}{x^2} \qquad f'(1)=-\frac{1}{1^2}=-1$$

$$f''(x)=2x^{-3}=\frac{2}{x^3} \qquad f''(1)=\frac{2}{1^3}=2$$

$$f'''(x)=-6x^{-4}=-\frac{6}{x^4} \qquad f'''(1)=-\frac{6}{1^4}=-6$$

$$f^{(4)}(x)=24x^{-5}=\frac{24}{x^5} \qquad f^{(4)}(1)=\frac{24}{1^5}=24$$

Substituting these values into the generic Taylor series (7.22) gives

$$f(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(2)}{3!}(x-1)^3+\ldots$$

$$=1+(-1)(x-1)+\frac{2}{2!}(x-1)^2+\frac{(-6)}{3!}(x-1)^3+\frac{24}{4!}(x-1)^4+\ldots$$

$$=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-\ldots \qquad \text{[Simplifying]}$$

(c) We need to find the Taylor series for $f(x)=e^x$ at $x=3$. Differentiating $f(x)=e^x$ and substituting $x=3$ into the resulting derivative gives:

$$f(x)=e^x \qquad f(3)=e^3$$

$$f'(x)=e^x \qquad f'(3)=e^3$$

$$f''(x)=e^x \qquad f''(3)=e^3$$

$$f'''(x)=e^x \qquad f'''(3)=e^3$$

$$f^{(4)}(x)=e^x \qquad f^{(4)}(3)=e^3$$

Putting these values into (7.22) gives

$$f(x)=f(3)+f'(3)(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3+\frac{f^{(4)}(3)}{4!}(x-3)^4+\ldots$$

$$=e^3+e^3(x-3)+e^3\frac{(x-3)^2}{2!}+e^3\frac{(x-3)}{3!}+e^3\frac{(x-3)^4}{4!}+\ldots$$

$$=e^3\left[1+(x-3)+\frac{(x-3)^2}{2!}+\frac{(x-3)}{3!}+\frac{(x-3)^4}{4!}+\ldots\right] \qquad \left[\begin{array}{l}\text{Taking Out the}\\ \text{common factor } e^3\end{array}\right]$$

(d) Very similar to Example 22. Let $f(x)=\cos(x)$ then we have

$$f(x)=\cos(x) \qquad f\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$f'(x)=-\sin(x) \qquad f'\left(\frac{\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$f''(x) = -\cos(x) \qquad f''\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = \sin(x) \qquad f'''\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f^{(4)}(x) = \cos(x) \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Substituting these values into (7.22) we have Taylor series for $\cos(x)$ at $x = \frac{\pi}{4}$:

$$\sin(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^4 + \ldots$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{\sqrt{2}}\frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{\sqrt{2}}\frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \ldots$$

$$= \frac{1}{\sqrt{2}}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \ldots\right] \qquad \left[\text{Taking out common factor } 1/\sqrt{2}\right]$$

(e) Similarly we find the Taylor series for $f(x) = \ln(x)$ at $x = e$:

$$f(x) = \ln(x) \qquad f(e) = \ln(e) = 1$$

$$f'(x) = \frac{1}{x} = x^{-1} \qquad f'(e) = \frac{1}{e}$$

$$f''(x) = -x^{-2} = -\frac{1}{x^2} \qquad f''(1) = -\frac{1}{e^2}$$

$$f'''(x) = 2x^{-3} = \frac{2}{x^3} \qquad f'''(1) = \frac{2}{e^3}$$

$$f^{(4)}(x) = -6x^{-4} = -\frac{6}{x^4} \qquad f^{(4)}(1) = -\frac{6}{e^4}$$

Substituting these values into the general Taylor series we have

$$f(x) = f(e) + f'(e)(x-e) + \frac{f''(e)}{2!}(x-e)^2 + \frac{f'''(e)}{3!}(x-e)^3 + \frac{f^{(4)}(e)}{4!}(x-e)^4 + \ldots$$

$$= 1 + \frac{1}{e}(x-e) + \left(\frac{-1}{e^2}\right)\frac{(x-e)^2}{2!} + \left(\frac{2}{e^3}\right)\frac{(x-e)^3}{3!} + \left(\frac{-6}{e^4}\right)\frac{(x-e)^4}{4!} + \ldots$$

$$= 1 + \frac{(x-e)}{e} - \frac{(x-e)^2}{2e^2} + \frac{(x-e)^3}{3e^3} - \frac{(x-e)^4}{4e^4} + \ldots \qquad [\text{Simplifying}]$$

7. We can replace x with $-\frac{1}{2}x^2$ into (7.15):

$$e^{-\frac{1}{2}x^2} = 1 + \left(-\frac{1}{2}x^2\right) + \frac{\left(-\frac{1}{2}x^2\right)^2}{2!} + \frac{\left(-\frac{1}{2}x^2\right)^3}{3!} + \ldots$$

$$\underbrace{=}_{\text{using the rules of indices}} 1 - \frac{1}{2}x^2 + \frac{\frac{1}{4}x^4}{2} + \frac{\left(-\frac{1}{8}\right)x^6}{6} + \ldots$$

$$= 1 - \frac{1}{2}x^2 + \frac{x^4}{8} - \frac{x^6}{48}\ldots$$

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8. We can apply the rules of logs to $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. By (5.12) we have

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

How can we find the Maclaurin series?
By using (7.21) and (7.22)

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\ldots\right)}_{\text{by (7.21)}} - \underbrace{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}\ldots\right)}_{\text{by (7.22)}}\right]$$

$$= \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\ldots\right) + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\ldots\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots\right]$$

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \qquad \text{(Cancelling 2's)}$$

(5.12) $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$

(7.15) $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

(7.21) $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$

(7.22) $\ln(1-x) = -x - x^2/2! - x^3/3! - x^4/4! - \ldots$