Complete solutions to Exercise 8(f)

1. (a) *Which formula do we use to write*

$$\frac{3x+4}{(x+1)(x+2)}$$

into partial fractions?

Use (8.48)

$$\frac{3x+4}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2} \qquad (\dagger)$$

Multiplying both sides by $(x+1)(x+2)$ gives:

$$3x+4=A(x+2)+B(x+1) \qquad (*)$$

How do we determine the constants A and B*?*

Put $x=-2$ into (*)

$$[3\times(-2)]+4=0+B(-2+1)$$

$$-2=-B \text{ gives } B=2$$

To find A, put $x=-1$ into (*)

$$[3\times(-1)]+4=A(-1+2)+0 \text{ gives } 1=A$$

Substituting $A=1$ and $B=2$ into $(\dagger)$ gives the partial fractions

$$\frac{3x+4}{(x+1)(x+2)}=\frac{1}{x+1}+\frac{2}{x+2}$$

(b) First we need to factorize the denominator:

$$t^2-1=(t-1)(t+1)$$

By applying (8.48) we have

$$\frac{2t}{t^2-1}=\frac{2t}{(t-1)(t+1)}=\frac{A}{t-1}+\frac{B}{t+1} \qquad (\dagger)$$

Multiplying both sides by $(t-1)(t+1)$:

$$2t=A(t+1)+B(t-1) \qquad (*)$$

Substituting $t=-1$ into (*) produces

$$2\times(-1)=0+B(-1-1)$$

$$-2=-2B \text{ gives } B=1$$

Substituting $t=1$ into (*) gives

$$(2\times1)=A(1+1)+0$$

$$2=2A$$

$$A=1$$

Putting $A=1$ and $B=1$ into $(\dagger)$

$$\frac{2t}{t^2-1}=\frac{1}{t-1}+\frac{1}{t+1}$$

(c) Factorizing the denominator:

$$s^2+s-2=(s+2)(s-1)$$

We have

(8.48) $$\frac{f(x)}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d}$$

$$\frac{2s+7}{s^2+s-2}=\frac{2s+7}{(s+2)(s-1)}\underbrace{=}_{\text{by (8.48)}}\frac{A}{s+2}+\frac{B}{s-1} \qquad (\dagger)$$

Thus

$$2s+7=A(s-1)+B(s+2) \qquad (*)$$

Need to find A and B. Put $s=1$ into (*)

$$2+7=0+B(1+2)$$

$$9=3B$$

$$B=3$$

Put $s=-2$ into (*):

$$[2\times(-2)]+7=A(-2-1)+0$$

$$3=-3A$$

$$A=-1$$

Substituting $A=-1$ and $B=3$ into $(\dagger)$ gives

$$\frac{2s+7}{s^2+s-2}=\frac{3}{s-1}-\frac{1}{s+2}$$

(d) We have

$$\frac{-12u-13}{(2u+1)(u-3)}=\frac{A}{2u+1}+\frac{B}{u-3} \qquad (\dagger)$$

Multiplying both sides by $(2u+1)(u-3)$:

$$-12u-13=A(u-3)+B(2u+1) \qquad (*)$$

Putting $u=3$ into (*):

$$(-12\times 3)-13=0+B[(2\times 3)+1]$$

$$-49=7B$$

$$B=-7$$

Putting $u=-\frac{1}{2}$ into (*) gives

$$6-13=A\left(-\frac{1}{2}-3\right)+0,\ -7=-\frac{7}{2}A \text{ gives } A=2$$

Substituting $A=2$ and $B=-7$ into $(\dagger)$:

$$\frac{-12u-13}{(2u+1)(u-3)}=\frac{2}{2u+1}-\frac{7}{u-3}$$

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2. (a) Since the degree of x^2 and $x+1$ is 2 and 1 respectively, we need to apply long division because it is an improper fraction.

$$\begin{array}{r|l} & \ \ x-1 \\ \hline x+1 & x^2 \\ & \underline{x^2+x} \\ & \quad -x \\ & \quad \underline{-x-1} \\ & \qquad 1 \end{array}$$

Using (8.52) with $q(x)=x-1$ and $R(x)=1$ we have

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

(b) The degree of $x^5 - 2x^2$ is 5 and degree of $x^2 - 1$ is 2.
Hence we need to employ long division.

$$\begin{array}{r l}
 & x^3 + x - 2 \\
x^2 - 1 \big) & x^5 - 2x^2 \\
 & \underline{x^5 - x^3} \\
 & \quad x^3 - 2x^2 \\
 & \quad \underline{x^3 - x} \\
 & \qquad -2x^2 + x \\
 & \qquad \underline{-2x^2 + 2} \\
 & \qquad\qquad x - 2
\end{array}$$

Applying (8.52) gives:

$$\frac{x^5 - 2x^2}{x^2 - 1} = x^3 + x - 2 + \frac{x-2}{x^2-1} \qquad (\dagger)$$

Is this our final answer or can we put $(\dagger)$ *into more partial fractions?*

Since $x^2 - 1$ factorizes we can place the last term, $\frac{x-2}{x^2-1}$, into partial fractions. We have

$$x^2 - 1 = x^2 - 1^2 = (x-1)(x+1)$$

Which formula, (8.48) - (8.51), do we use to write

$$\frac{x-2}{x^2-1} = \frac{x-2}{(x-1)(x+1)}$$

into partial fractions?

Use (8.48)

$$\frac{x-2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \qquad (\dagger\dagger)$$

Multiplying both sides by $(x-1)(x+1)$:

$$x - 2 = A(x+1) + B(x-1) \qquad (*)$$

Substituting $x = 1$ into (*):

$$1 - 2 = A(1+1) + B(0)$$

$$-1 = 2A \qquad \text{gives} \qquad A = -\frac{1}{2}$$

Substituting $x = -1$ into (*):

$$-1 - 2 = 0 + B(-1-1)$$

$$-3 = -2B \text{ gives } B = \frac{3}{2}$$

(8.48) $$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

(8.52) $$\frac{f(x)}{g(x)} = q(x) + \frac{R(x)}{g(x)}$$

Putting these into $(\dagger\dagger)$ gives:

$$\frac{x-2}{(x-1)(x+1)}=\frac{-\frac{1}{2}}{x-1}+\frac{\frac{3}{2}}{x+1}=\frac{1}{2}\left[\frac{3}{x+1}-\frac{1}{x-1}\right]$$

Substituting into $(\dagger)$:

$$\frac{x^5-2x^2}{x^2-1}=x^3+x-2+\frac{1}{2}\left[\frac{3}{x+1}-\frac{1}{x-1}\right]$$

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3. (a) *Which formula do we use to put*

$$\frac{4t^2+t-3}{(t^2+t-1)(t-1)}$$

into partial fractions?
(8.51)

$$\frac{4t^2+t-3}{(t^2+t-1)(t-1)}=\frac{At+B}{t^2+t-1}+\frac{C}{t-1} \qquad (\dagger)$$

Multiplying both sides by $(t^2+t-1)(t-1)$ gives:

$$4t^2+t-3=(At+B)(t-1)+C(t^2+t-1) \qquad (*)$$

Is this our final solution?
No, we need to find A, B and C. Putting $t=1$ into (*) gives

$$4+1-3=0+C(1^2+1-1)$$

$$2=C$$

How can we find A and B?
Equate coefficients of t^2. (The number of t^2 on the Left Hand Side of the = sign in (*) is same as the number of t^2 on the Right Hand Side of the = sign).

$$4=A+C$$

We already know $C=2$, so

$$4=A+2$$

$$A=2$$

Next we equate coefficients of t:

$$1=\underbrace{-A+B+C}_{\text{This is obtained by expanding the brackets in (*)}}$$

We know $A=2$ and $C=2$, so

$$1=-2+B+2$$

$$B=1$$

Substituting $A=2$, $B=1$ and $C=2$ into $(\dagger)$ gives

$$\frac{4t^2+t-3}{(t^2+t-1)(t-1)}=\frac{2t+1}{t^2+t-1}+\frac{2}{t-1}$$

(b) By using (8.49) we have

(8.49) $$\frac{f(z)}{(az+b)^2}=\frac{A}{az+b}+\frac{B}{(az+b)^2}$$

(8.51) $$\frac{f(t)}{(at^2+bt+c)(dt+e)}=\frac{At+B}{at^2+bt+c}+\frac{C}{dt+e}$$

$$\frac{z+1}{(z-1)^2}=\frac{A}{z-1}+\frac{B}{(z-1)^2} \qquad (\dagger)$$

Multiplying $(\dagger)$ by $(z-1)^2$ gives

$$z+1=A(z-1)+B \qquad (*)$$

Putting $z=1$ into (*) gives

$$2=0+B$$

$$B=2$$

Equating coefficients of z in (*) gives

$$1=A$$

Substituting $A=1$ and $B=2$ into $(\dagger)$:

$$\frac{z+1}{(z-1)^2}=\frac{1}{z-1}+\frac{2}{(z-1)^2}$$

(c) *Which formula do we use to put*

$$\frac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$$

into partial fractions?
Use repeated factor

$$\frac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2} \qquad (\dagger)$$

Multiplying both sides of $(\dagger)$ by $(x^2+x+1)^2$ we have

$$2x^3+3x^2+5x+2=(Ax+B)(x^2+x+1)+(Cx+D)$$

$$=Ax^3+Ax^2+Ax+Bx^2+Bx+B+(Cx+D)$$

$$2x^3+3x^2+5x+2=Ax^3+(A+B)x^2+(A+B+C)x+B+D \qquad (*)$$

How do we find A, B, C and D?
We can equate coefficients of powers of x. Consider the highest power first, x^3.
There are 2 on the left and A on the right

$$2=A$$

How many x^2 *on the left of the = sign in (*)?*

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The number of x^2 on the right of the = sign in (*) is ______

$$A+B$$

Thus equating coefficients of x^2 gives

$$3=A+B$$

We know $A=2$, so $B=1$.
Next we equate coefficients of x in (*):

$$5=A+B+C$$

Putting $A=2$ and $B=1$ gives

$$5=2+1+C$$

$$C=2$$

Equating coefficients of constants in (*):

$$2=B+D$$

$B=1$ so $D=1$
Substituting $A=2$, $B=1$, $C=2$ and $D=1$ into $(\dagger)$ gives the partial fractions:

$$\frac{2x^3+3x^2+5x+2}{\left(x^2+x+1\right)^2}=\frac{2x+1}{x^2+x+1}+\frac{2x+1}{\left(x^2+x+1\right)^2}$$