Complete solutions to Exercise 8(g)

1. All the functions that need to be integrated are given in question 1 of EXERCISE 8(f) with a different variable. So we can use the partial fractions already established.

(a) $\int \frac{3c+4}{(c+1)(c+2)}dc \underbrace{=}_{\text{by solution 1(a)}} \int\left(\frac{1}{c+1}+\frac{2}{c+2}\right)dc$

$$= \int \frac{dc}{c+1} + 2\int \frac{dc}{c+2} \underbrace{=}_{\text{by (8.42)}} \ln|c+1| + 2\ln|c+2| + C$$

(b) $\int \frac{2\lambda}{\lambda^2-1}d\lambda = \underbrace{\ln|\lambda^2-1|}_{\text{by (8.42)}} + C$. We don't need to write $\frac{2\lambda}{\lambda^2-1}$ into partial fractions because the derivative of the denominator = numerator.

(c) $\int \frac{2a+7}{a^2+a-2}da = \int\left(\frac{3}{a-1}-\frac{1}{a+2}\right)da \quad \begin{pmatrix}\text{By Solution 1(c)}\\ \text{of Exercise 8(f)}\end{pmatrix}$

$$= 3\int \frac{da}{a-1} - \int \frac{da}{a+2}$$

$$= 3\ln|a-1| - \ln|a+2| + C$$

$$= \underbrace{\ln|a-1|^3}_{\text{by (5.13)}} - \ln|a+2| + C = \underbrace{\ln\left|\frac{(a-1)^3}{a+2}\right|}_{\text{by (5.12)}} + C$$

(d) By solution 1(d) of Exercise 8(f)

$$\int \frac{-12y-13}{(2y+1)(y-3)}dy = \int\left(\frac{2}{2y+1}-\frac{7}{y-3}\right)dy$$

$$= \int \frac{2dy}{2y+1} - 7\int \frac{dy}{y-3}$$

$$\underbrace{=}_{\text{by (8.42)}} \ln|2y+1| - 7\ln|y-3| + C$$

$$= \ln|2y+1| - \underbrace{\ln|y-3|^7}_{\text{by (5.13)}} + C = \underbrace{\ln\left|\frac{2y+1}{(y-3)^7}\right|}_{\text{by (5.12)}} + C$$

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2. We use solutions to question 3 Exercise 8(f).

(a) $\int \frac{4p^2+p-3}{(p^2+p-1)(p-1)}dp \underbrace{=}_{\text{by solution 3(a)}} \int\left(\frac{2p+1}{p^2+p-1}+\frac{2}{p-1}\right)dp$

$$= \int \frac{2p+1}{p^2+p-1}dp + 2\int \frac{dp}{p-1}$$

$$= \ln|p^2+p-1| + 2\ln|p-1| + C \quad (\text{by (8.42)})$$

$$= \ln|p^2+p-1| + \underbrace{\ln(p-1)^2}_{\text{by (5.13)}} + C$$

(5.12) $\ln(A) - \ln(B) = \ln(A/B)$

(5.13) $n\ln(A) = \ln(A^n)$

(8.42) $\int f'(x)/f(x)dx = \ln|f(x)|$

(b) By solution of question 3(b):

$$\int \frac{z+1}{(z-1)^2}dz = \int\left(\frac{1}{z-1}+\frac{2}{(z-1)^2}\right)dz$$
$$= \int \frac{dz}{z-1} + 2\int \frac{dz}{(z-1)^2}$$

The first integral, $\int \frac{dz}{z-1}$ *, is uncomplicated but how do we find* $\int \frac{dz}{(z-1)^2}$ *?*

Let $u = z-1$, then differentiation gives

$$\frac{du}{dz} = 1 \text{ it follows that } du = dz$$

Thus we have

$$\int \frac{dz}{(z-1)^2} = \int \frac{du}{u^2} = \int u^{-2}du = \frac{u^{-1}}{-1}$$
$$= -u^{-1} = -\frac{1}{u} = -\frac{1}{z-1} = \frac{1}{1-z}$$

Of course there is a constant but it is easier to add this on at the end. So we have

$$\int \frac{z+1}{(z-1)^2}dz = \int \frac{dz}{z-1} + \frac{2}{1-z}$$
$$= \ln|z-1| + \frac{2}{1-z} + C$$

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3. First we put $\frac{5z^2}{(z^2+1)(2z-1)}$ into partial fractions. By (8.51) we have

$$\frac{5z^2}{(z^2+1)(2z-1)} = \frac{Az+B}{z^2+1} + \frac{C}{2z-1} \qquad (\dagger)$$

Multiply both sides by $(z^2+1)(2z-1)$:

$$5z^2 = (Az+B)(2z-1) + C(z^2+1) \qquad (*)$$

Putting $z = \frac{1}{2}$ into (*) gives:

$$5\left(\frac{1}{4}\right) = 0 + C\left(\frac{1}{4}+1\right)$$
$$\frac{5}{4} = \frac{5}{4}C \text{ gives } C = 1$$

How can we find A and B ?

Equating coefficients of z^2 - the number of z^2 on the left of the = sign in (*) is equal to the number of z^2 on the right of the = sign in (*). Thus

$$5 = 2A + C$$

We already have $C = 1$, so

$$5 = 2A + 1,\ \ 4 = 2A \text{ gives } A = 2$$

How can we find B ?

(8.51) $$\frac{f(z)}{(az^2+bz+c)(dz+e)} = \frac{Az+B}{az^2+bz+c} + \frac{C}{dz+e}$$

Equate coefficients of z's. *How many z's are there on the left of the = sign in (*)?*

$$0$$

How many z's are there on the right of the = sign in (*)?

$$-A+2B$$
$$0=-A+2B$$

We know $A=2$,

$$0=-2+2B,\ 2=2B \text{ gives } B=1$$

Substituting $A=2,\ B=1$ and $C=1$ into $(\dagger)$ gives

$$\frac{5z^2}{(z^2+1)(2z-1)}=\frac{2z+1}{z^2+1}+\frac{1}{2z-1}$$

The integral becomes

$$\int_1^2 \frac{5z^2}{(z^2+1)(2z-1)}dz=\int_1^2 \frac{2z+1}{z^2+1}dz+\int_1^2 \frac{dz}{2z-1}$$

The second integral on the RHS, $\int \frac{dz}{2z-1}$, is straightforward but how do we integrate $\frac{2z+1}{z^2+1}$?

This can be broken into

$$\frac{2z+1}{z^2+1}=\frac{2z}{z^2+1}+\frac{1}{z^2+1}$$

So we have

$$\int_1^2 \frac{5z^2}{(z^2+1)(2z-1)}dz=\int_1^2 \frac{2z}{z^2+1}dz+\int_1^2 \frac{dz}{z^2+1}+\int_1^2 \frac{dz}{2z-1}$$

$$=\left[\ln\left|z^2+1\right|\right]_1^2+\underbrace{\left[\tan^{-1}(z)\right]_1^2}_{\text{by (8.26)}}+\frac{1}{2}\left[\ln\left|2z-1\right|\right]_1^2$$

$$=\left[\ln(5)-\ln(2)\right]+\left[\tan^{-1}(2)-\tan^{-1}(1)\right]+\frac{1}{2}\left[\ln(3)-\ln(1)\right]=1.79$$

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4. First we place $\frac{1}{v(2v+1)}$ into partial fractions. *Which formula do we use?*

By (8.48):

$$\frac{1}{v(2v+1)}=\frac{A}{v}+\frac{B}{2v+1} \qquad (\dagger)$$

Multiplying both sides by $v(2v+1)$:

$$1=A(2v+1)+Bv \qquad (*)$$

Substituting $v=0$ into (*): $1=A$

Substituting $v=-1/2$ into (*): $1=0+B\left(-\frac{1}{2}\right)=-\frac{1}{2}B$ gives $B=-2$

(8.26) $$\int \frac{dz}{a^2+z^2}=\frac{1}{a}\tan^{-1}\left(\frac{z}{a}\right)$$

(8.48) $$\frac{f(v)}{(av+b)(cv+d)}=\frac{A}{av+b}+\frac{B}{cv+d}$$

Putting $A=1$ and $B=-2$ into $(\dagger)$ gives:

$$\frac{1}{v(2v+1)}=\frac{1}{v}-\frac{2}{2v+1}$$

We have

$$t=\int_{10}^{100}\frac{dv}{v(2v+1)}=\int_{10}^{100}\left(\frac{1}{v}-\frac{2}{2v+1}\right)dv$$

$$=\int_{10}^{100}\frac{dv}{v}-\int_{10}^{100}\frac{2dv}{2v+1}$$

$$=\Big[\ln|v|-\ln|2v+1|\Big]_{10}^{100}\quad\left(\text{by (8.42)}\right)$$

$$=\big[\ln(100)-\ln(201)\big]-\big[\ln(10)-\ln(21)\big]=0.044$$

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5. The function $\frac{1}{1-(kv)^2}$ can be written in partial fractions. First we factorize the denominator,

$$1-(kv)^2=(1+kv)(1-kv)$$

So we have

$$\frac{1}{1-(kv)^2}=\frac{1}{(1+kv)(1-kv)}$$

How do we write $\frac{1}{(1+kv)(1-kv)}$ *into partial fractions?*

Use (8.48)

$$\frac{1}{(1+kv)(1-kv)}=\frac{A}{1+kv}+\frac{B}{1-kv}\qquad(\dagger)$$

Multiply both sides by $(1+kv)(1-kv)$:

$$1=A(1-kv)+B(1+kv)\qquad(*)$$

How do we find A and B*?*
Substitute $v=1/k$ into (*):

$$1=0+B\left(1+k\frac{1}{k}\right),\ 1=2B\ \text{ gives }\ B=\frac{1}{2}$$

Similarly by putting $v=-1/k$ into (*) gives

$$1=A\left[1-k\left(-\frac{1}{k}\right)\right]+0=A\left[1+k\left(\frac{1}{k}\right)\right]$$

$$1=2A \text{ gives } A=1/2$$

Putting $A=1/2$ and $B=1/2$ into $(\dagger)$ gives the partial fractions

$$\frac{1}{(1+kv)(1-kv)}=\frac{1/2}{1+kv}+\frac{1/2}{1-kv}$$

$$=\frac{1}{2}\left[\frac{1}{1+kv}+\frac{1}{1-kv}\right]\quad\left(\begin{array}{l}\text{Taking out the}\\ \text{Common Factor } 1/2\end{array}\right)$$

(8.42) $\quad\int f'(x)/f(x)dx=\ln|f(x)|$

(8.48) $\quad\dfrac{f(v)}{(av+b)(cv+d)}=\dfrac{A}{av+b}+\dfrac{B}{cv+d}$

The integral becomes

$$\int \frac{dv}{1-(kv)^2} = \frac{1}{2}\left[\int \frac{dv}{1+kv} + \int \frac{dv}{1-kv}\right] \qquad (\dagger\dagger)$$

How do we find $\int \frac{dv}{1+kv}$ *?*

Use (8.42). Thus

$$\int \frac{dv}{1+kv} = \frac{1}{k}\ln|1+kv|$$

There is a constant of integration but we add this to the final term. Similarly

$$\int \frac{dv}{1-kv} = -\frac{1}{k}\ln|1-kv|$$

Using $(\dagger\dagger)$ and adding the constant C gives

$$\int \frac{dv}{1-(kv)^2} = \frac{1}{2}\left[\frac{1}{k}\ln|1+kv| - \frac{1}{k}\ln|1-kv|\right] + C$$

$$= \frac{1}{2k}\left[\ln|1+kv| - \ln|1-kv|\right] + C = \frac{1}{2k}\ln\left|\frac{1+kv}{1-kv}\right| + C$$

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6. First we express $\frac{5+2x-x^2}{(x^2+1)(x+1)}$ into partial fractions. By (8.50) we have

$$\frac{5+2x-x^2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} \qquad (\dagger)$$

Multiplying both sides of $(\dagger)$ by $(x^2+1)(x+1)$ gives

$$5+2x-x^2 = (Ax+B)(x+1) + C(x^2+1) \qquad (*)$$

What do we need to determine next?

The constants A, B and C. *What values of* x *should we substitute into (*)?*

Put $x=-1$ to remove the first term on the RHS of (*)

$$5+[2\times(-1)]-(-1)^2 = 0 + C\left((-1)^2+1\right), \quad 2 = 2C \text{ gives } C = 1$$

How can we find A and B*?*

We need to equate coefficients of x^2 in (*):

$$-1 = A + C = A + 1 \text{ gives } A = -2$$

Equating coefficients of x in (*):

$$2 = A + B = -2 + B \text{ gives } B = 4$$

Substituting $A=-2$, $B=4$ and $C=1$ into $(\dagger)$:

$$\frac{5+2x-x^2}{(x^2+1)(x+1)} = \frac{-2x+4}{x^2+1} + \frac{1}{x+1}$$

$$= \frac{4-2x}{x^2+1} + \frac{1}{x+1}$$

We have

(8.42) $\int f'(x)/f(x)dx = \ln|f(x)|$

(8.50) $\frac{f(x)}{(ax^2+bx+c)(dx+e)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$

$$\int_0^1 \frac{5+2x-x^2}{(x^2+1)(x+1)}dx = \int_0^1 \left(\frac{4-2x}{x^2+1}+\frac{1}{x+1}\right)dx$$

$$= \int_0^1 \left(\frac{4-2x}{x^2+1}\right)dx + \int_0^1 \frac{dx}{x+1} \qquad (\dagger\dagger)$$

Since we cannot find the integral of $\frac{4-2x}{x^2+1}$, this can be broken into

$$\frac{4-2x}{x^2+1} = \frac{4}{x^2+1} - \frac{2x}{x^2+1}$$

Substituting this into $(\dagger\dagger)$ gives

$$\int_0^1 \frac{5+2x-x^2}{(x^2+1)(x+1)}dx = \int_0^1 \frac{4dx}{x^2+1} - \int_0^1 \frac{2x}{x^2+1}dx + \int_0^1 \frac{dx}{x+1}$$

$$= 4\underbrace{\left[\tan^{-1}(x)\right]_0^1}_{\text{by (8.26)}} - \underbrace{\left[\ln\left|x^2+1\right|\right]_0^1}_{\text{by (8.42)}} + \underbrace{\left[\ln\left|x+1\right|\right]_0^1}_{\text{by (8.42)}}$$

$$= 4\left[\tan^{-1}(x)\right]_0^1 + \left[\ln|x+1| - \ln\left|x^2+1\right|\right]_0^1$$

$$= 4\left[\tan^{-1}(1) - \tan^{-1}(0)\right] + \left[\ln(2) - \ln(2) - \left(\ln(1) - \ln(1)\right)\right]$$

$$= 4\tan^{-1}(1)$$

$$= 4\left(\frac{\pi}{4}\right) = \pi$$

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7. (i) Note that in $\frac{x^3+1}{x^2+3x+2}$, the numerator, x^3+1, is a higher degree polynomial than the denominator, x^2+3x+2. So we need to first divide out by long division. Thus

$$\begin{array}{r} x-3 \\ x^2+3x+2\overline{)\,x^3+1} \\ \underline{x^3+3x^2+2x} \\ 1-3x^2-2x \\ \underline{-3x^2-9x-6} \\ 0+7x+7 \end{array}$$

We have

$$\frac{x^3+1}{x^2+3x+2} = x-3+\frac{7x+7}{x^2+3x+2} \qquad (*)$$

We need to put $\frac{7x+7}{x^2+3x+2}$ into partial fractions.

$$x^2+3x+2 = (x+2)(x+1)$$

Thus

(8.26) $$\int \frac{dz}{a^2+z^2} = \frac{1}{a}\tan^{-1}\left(\frac{z}{a}\right)$$

(8.42) $$\int \left[\frac{f'(x)}{f(x)}\right]dx = \ln|f(x)|$$

$$\frac{7x+7}{x^2+3x+2}=\frac{7x+7}{(x+2)(x+1)}$$
$$=\frac{7(x+1)}{(x+2)(x+1)}$$
$$=\frac{7}{x+2}$$

Therefore $\frac{7x+7}{x^2+3x+2}$ is a single fraction $\frac{7}{x+2}$. Substituting this into (*) gives

$$\frac{x^3+1}{x^2+3x+2}=x-3+\frac{7}{x+2}$$

(ii) We have

$$\int\frac{x^3+1}{x^2+3x+2}dx=\int\left(x-3+\frac{7}{x+2}\right)dx$$
$$=\int(x-3)dx+7\int\frac{dx}{x+2}$$
$$=\frac{x^2}{2}-3x+\underbrace{7\ln|x+2|}_{\text{by (8.42)}}+C$$

(8.42) $$\int\left[\frac{f'(x)}{f(x)}\right]dx=\ln|f(x)|$$