Complete solutions to Exercise 8(h)

1. (a) *Which substitution should we use to find* $\int b\left(b^2-3\right)^7 db$ *?*

Let

$$u=b^2-3,\ \frac{du}{db}=2b$$

We need to replace db.

$$db=\frac{du}{2b}$$

We have

$$\int b\left(b^2-3\right)^7 db=\int bu^7\frac{du}{2b}=\frac{1}{2}\int u^7du=\underbrace{\frac{1}{2}\left(\frac{u^8}{8}\right)+C}_{\text{by (8.1)}}$$

$$=\frac{\left(b^2-3\right)^8}{16}+C\quad\left(\text{Remember } u=b^2-3\right)$$

(b) Let $u=5s-1$, then

$$\frac{du}{ds}=5\text{ gives } ds=\frac{du}{5}$$

Substituting $u=5s-1$ and $ds=\dfrac{du}{5}$ gives

$$\int\left(5s-1\right)^9ds=\int u^9\frac{du}{5}\qquad\left[\text{Substituting}\right]$$

$$=\frac{1}{5}\int u^9du$$

$$=\frac{1}{5}\left(\frac{u^{10}}{10}\right)+C$$

$$=\frac{u^{10}}{50}+C$$

$$=\frac{\left(5s-1\right)^{10}}{50}+C$$

(c) Differentiating a^3-2a^2+6 with respect to a gives $3a^2-4a$, which is the first term in the bracket. So use $u=a^3-2a^2+6$,

$$\frac{du}{da}=3a^2-4a$$

$$da=\frac{du}{3a^2-4a}$$

Using these substitutions we have

$$\int\left(3a^2-4a\right)u^4\frac{du}{\left(3a^2-4a\right)}=\int u^4du$$

$$=\frac{u^5}{5}+C=\frac{\left(a^3-2a^2+6\right)^5}{5}+C$$

(8.1) $\int u^n du=u^{n+1}/n+1$

(d) The integral $\int 21q^2\sqrt{7q^3-5}\,dq$ is of the format (8.53). *Which substitution should we use?*
Let

$$u = 7q^3 - 5$$

then

$$\frac{du}{dq} = 21q^2 \text{ gives } dq = \frac{du}{21q^2}$$

Putting $u = 7q^3 - 5$ and $dq = \dfrac{du}{21q^2}$ into the original integral gives

$$\int 21q^2\sqrt{7q^3-5}\,dq = \int 21q^2\sqrt{u}.\frac{du}{21q^2}$$

$$= \int \sqrt{u}\,du = \int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{3/2} + C = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2\left(7q^3-5\right)^{\frac{3}{2}}}{3} + C$$

(e) To find $\int \dfrac{p^2-1}{\sqrt{p^3-3p}}dp$ we can use the substitution $u = p^3 - 3p$. Then

$$\frac{du}{dp} = 3p^2 - 3$$

$$dp = \frac{du}{3p^2-3} = \frac{du}{3\left(p^2-1\right)} \qquad \left[\begin{array}{l}\text{Taking Out a}\\ \text{Common Factor of 3}\end{array}\right]$$

Putting $u = p^3 - 3p$ and $dp = \dfrac{du}{3\left(p^2-1\right)}$ gives

$$\int\left(\frac{p^2-1}{\sqrt{p^3-3p}}\right)dp = \int\left(\frac{p^2-1}{\sqrt{u}}\right)\frac{du}{3\left(p^2-1\right)}$$

$$= \frac{1}{3}\int\frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-\frac{1}{2}}du = \frac{1}{3}\left[\frac{u^{1/2}}{1/2}\right] + C = \frac{2}{3}u^{1/2} + C$$

$$\int \frac{p^2-1}{\sqrt{p^3-3p}}dp = \frac{2}{3}\left(p^2-3p\right)^{1/2} + C$$

(f) Let $u = \alpha^2 - 2\alpha + 10$ then

$$\frac{du}{d\alpha} = 2\alpha - 2 \text{ gives } d\alpha = \frac{du}{2\left(\alpha-1\right)}$$

We have

$$\int\left(\frac{\alpha-1}{\left(\alpha^2-2\alpha+10\right)^2}\right)d\alpha = \int\left(\frac{\alpha-1}{u^2}\right)\frac{du}{2\left(\alpha-1\right)}$$

(8.1) $$\int\left(u^n\right)du = \frac{u^{n+1}}{n+1}$$

$$=\frac{1}{2}\int\frac{du}{u^2}$$

$$=\frac{1}{2}\int u^{-2}du$$

$$=\underbrace{\frac{1}{2}\left[\frac{u^{-1}}{-1}\right]}_{\text{by (8.1)}}+C=-\frac{1}{2}u^{-1}+C=-\frac{1}{2}\left(\alpha^2-2\alpha+10\right)^{-1}+C$$

2. Let $u=1-0.2t$ then

$$\frac{du}{dt}=-0.2$$

$$dt=\frac{du}{-0.2} \text{ gives } dt=-5du$$

We also need to replace the limits:

$$t=0,\quad u=\left[1-\left(0.2\times0\right)\right]=1$$

$$t=5,\quad u=\left[1-\left(\underbrace{0.2\times5}_{=1}\right)\right]=0$$

We have

$$MTTF=\int_0^5\left(1-0.2t\right)^{1.5}dt$$

$$=\int_1^0 u^{1.5}\left(-5du\right)$$

$$=-5\int_1^0 u^{1.5}du=-5\left[\frac{u^{2.5}}{2.5}\right]_1^0=-5\left[\frac{0-1}{2.5}\right]=2$$

So $MTTF=2$ years .

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3. Let $u=x^2$ then

$$\frac{du}{dx}=2x \text{ which gives } \quad dx=\frac{du}{2x}$$

We also need to replace the limits of integration:

$$x=0,\qquad u=0^2=0$$

$$x=1,\qquad u=1^2=1$$

We have

$$\int_{x=0}^{x=1}\left(xe^{-x^2}\right)dx=\int_{u=0}^{u=1}\left(xe^{-u}\right)\frac{du}{2x}$$

$$=\frac{1}{2}\int_0^1 e^{-u}du \qquad \left(\text{Cancelling } x\right)$$

$$=\frac{1}{2}\left[-e^{-u}\right]_0^1=-\frac{1}{2}\left(e^{-1}-\underbrace{e^0}_{=1}\right)=0.316$$

4. (a) *Which substitution should we employ?*
Let $u = \cos(\theta)$ then

$$\frac{du}{d\theta} = -\sin(\theta)$$

$$d\theta = -\frac{du}{\sin(\theta)}$$

We also need to replace the limits. Thus
When $\theta = 0,\ u = \cos(0) = 1$
When $\theta = \frac{\pi}{2},\ u = \cos\left(\frac{\pi}{2}\right) = 0$
We have

$$\int_{\theta=0}^{\theta=\pi/2} \sin(\theta)\sqrt{\cos(\theta)}d\theta = \int_{u=1}^{u=0} \sin(\theta)\sqrt{u}.\left(\frac{-du}{\sin(\theta)}\right)$$

$$= -\int_1^0 \sqrt{u}du \qquad \left(\text{Cancelling } \sin(\theta)\right)$$

$$= -\int_1^0 u^{1/2}du$$

$$= -\left[\frac{u^{3/2}}{3/2}\right]_1^0 = -\frac{2}{3}\left[u^{3/2}\right]_1^0 = -\frac{2}{3}\left[0^{3/2} - 1^{3/2}\right] = \frac{2}{3}$$

$$\int_0^{\pi/2} \sin(\theta)\sqrt{\cos(\theta)}d\theta = \frac{2}{3}$$

The integrals $\int_0^{\pi/2} \sin(\theta)\sqrt{\cos(\theta)}d\theta = \underbrace{\int_0^{\pi/2} \cos(\theta)\sqrt{\sin(\theta)}d\theta}_{\text{Evaluated in EXAMPLE 33}} = \frac{2}{3}$

(b) Let $u = \cos(\theta)$ then differentiating gives:

$$\frac{du}{d\theta} = -\sin(\theta) \text{ which gives } d\theta = -\frac{du}{\sin(\theta)}$$

Replacing limits into $u = \cos(\theta)$:
When $\theta = 0,\ u = \cos(0) = 1$
When $\theta = \pi,\ u = \cos(\pi) = -1$
We have

$$\int_{\theta=0}^{\theta=\pi} \sin(\theta)\cos^5(\theta)\,d\theta = \int_{u=1}^{u=-1} \sin(\theta)u^5\left(-\frac{du}{\sin(\theta)}\right)$$

$$= -\int_1^{-1} u^5 du = -\left[\frac{u^6}{6}\right]_1^{-1} = -\frac{1}{6}\underbrace{\left[(-1)^6 - 1^6\right]}_{=0} = 0$$

$$\int_0^{\pi} \sin(\theta)\cos^5(\theta)\,d\theta = 0$$

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5. (a) *Which substitution can we use?*
Let $u = \sec(\beta)$ then by (6.22):

$$\frac{du}{d\beta} = \sec(\beta)\tan(\beta)$$

(6.22) $\left[\sec(x)\right]' = \sec(x)\tan(x)$

We have

$$d\beta = \frac{du}{\sec(\beta)\tan(\beta)} \underbrace{=}_{\substack{\text{because} \\ u=\sec(\beta)}} \frac{du}{u\tan(\beta)}$$

Substituting $u = \sec(\beta)$ and $d\beta = \dfrac{du}{u\tan(\beta)}$ into

$$\int \sec^7(\beta)\tan(\beta)\,d\beta$$

gives

$$\begin{aligned}
\int u^7 \tan(\beta)\frac{du}{u\tan(\beta)} &= \int u^6 du \\
&= \frac{u^7}{7} + C \\
&= \frac{[\sec(\beta)]^7}{7} + C \\
&= \frac{\sec^7(\beta)}{7} + C
\end{aligned}$$

(b) Let $u = \tan(A)$ then

$$\frac{du}{dA} = \sec^2(A) \text{ which gives } dA = \frac{du}{\sec^2(A)}$$

Substituting $u = \tan(A)$ and $dA = \dfrac{du}{\sec^2(A)}$ produces:

$$\int \tan^5(A)\sec^2(A)\,dA = \int u^5 \sec^2(A)\frac{du}{\sec^2(A)}$$

$$= \int u^5 du = \frac{u^6}{6} + C = \frac{[\tan(A)]^6}{6} + C = \frac{\tan^6(A)}{6} + C$$

(c) If you use $u = \cot(A)$ the integration is lengthy whilst if you consider $u = \operatorname{cosec}(A)$ the solution is obtained in a few lines. So let $u = \operatorname{cosec}(A)$

$$\frac{du}{dA} \underbrace{=}_{\text{by (6.23)}} -\operatorname{cosec}(A)\cot(A), \text{ therefore } dA = -\frac{du}{\operatorname{cosec}(A)\cot(A)}$$

Substituting these we have

$$\int \cot^3(A)\operatorname{cosec}(A)\,dA = \int \cot^3(A)\operatorname{cosec}(A)\left(-\frac{du}{\operatorname{cosec}(A)\cot(A)}\right)$$

$$\begin{aligned}
&= -\int \cot^2(A)\,du \\
&= -\int \underbrace{\left(\operatorname{cosec}^2(A) - 1\right)du}_{\text{by (4.66)}} \\
&= -\int \left(u^2 - 1\right)du \qquad \left(\text{Because } u = \operatorname{cosec}(A)\right) \\
&= -\left(\frac{u^3}{3} - u\right) + C = u - \frac{u^3}{3} + C = \operatorname{cosec}(A) - \frac{\operatorname{cosec}^3(A)}{3} + C
\end{aligned}$$

(4.66) $1 + \cot^2(A) = \operatorname{cosec}^2(A)$ (6.23) $[\operatorname{cosec}(A)]' = -\operatorname{cosec}(A)\cot(A)$