Complete solutions to Exercise 8(i)

1. See solution to **EXAMPLE 17** on pages 425- 426.
2. We first evaluate the integral under the square root:

$$\int_0^{\frac{2\pi}{\omega}} v^2 dt = \int_0^{\frac{2\pi}{\omega}} \left[10\sin(\omega t)\right]^2 dt$$

$$= \int_0^{\frac{2\pi}{\omega}} 100\sin^2(\omega t)\,dt$$

$$= 100\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t)\,dt \qquad (*)$$

From solution to **EXAMPLE 17** (or see solution 3 below) we have:

$$\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t)\,dt = \frac{\pi}{\omega}$$

Putting this into (*) gives:

$$\int_0^{\frac{2\pi}{\omega}} v^2\,dt = \frac{100\pi}{\omega}$$

Substituting this into $V_{R.M.S.}$ gives:

$$V_{R.M.S.} = \sqrt{\frac{\omega}{2\pi}\frac{100\pi}{\omega}}$$

$$= \sqrt{\frac{\not\omega}{2\not\pi}\frac{100\not\pi}{\not\omega}}$$

$$= \sqrt{\frac{100}{2}} = \frac{\sqrt{100}}{\sqrt{2}} = \frac{10}{\sqrt{2}}\text{ V}$$

---

3. We have

$$P = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V\sin(\omega t) I\sin(\omega t)\,dt$$

$$= \frac{\omega VI}{2\pi}\int_0^{2\pi/\omega} \sin^2(\omega t)dt$$

$$= \frac{\omega VI}{2\pi}\int_0^{2\pi/\omega} \underbrace{\frac{1}{2}\left[1-\cos(2\omega t)\right]}_{\text{by (4.68)}}dt$$

$$= \frac{\omega VI}{4\pi}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_0^{2\pi/\omega} \qquad \left(\text{By } \int\cos(kt)\,dt = \frac{\sin(kt)}{k}\right)$$

$$= \frac{\omega VI}{4\pi}\left(\frac{2\pi}{\omega} - \frac{\sin(2\omega(2\pi/\omega))}{2\omega}\right) = \frac{\omega VI}{4\pi}\left(\frac{2\pi}{\omega} - 0\right) = \frac{VI}{2}$$

(4.68) $$\sin^2(A) = \frac{1}{2}\left[1-\cos(2A)\right]$$

4. Substituting $V=\cos(t)-\sin(t)$ and $2L = 2\times 10^{-3}$ into W gives:

$$W = \frac{1}{2\times10^{-3}}\left[\int_0^t\left[\cos(t)-\sin(t)\right]dt+1\right]^2$$
$$= \frac{1}{2\times10^{-3}}\left\{\left[\sin(t)+\cos(t)\right]_0^t+1\right\}^2$$
$$= \frac{1}{2\times10^{-3}}\left(\sin(t)+\cos(t)-1+1\right)^2$$
$$= \frac{1}{2\times10^{-3}}\left(\sin^2(t)+\underbrace{2\sin(t)\cos(t)}_{=\sin(2t)}+\cos^2(t)\right)$$
$$= \underbrace{\frac{1}{2\times10^{-3}}}_{=500}\left(1+\sin(2t)\right)=500\left(1+\sin(2t)\right)$$

---

5. Substituting $v = V\sin(\omega t)$ and $i = I\sin(\omega t+\phi)$ into P gives:

$$P = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}V\sin(\omega t)\,I\sin(\omega t+\phi)dt$$
$$= \frac{\omega VI}{2\pi}\int_0^{2\pi/\omega}\sin(\omega t)\sin(\omega t+\phi)dt \qquad (*)$$

How do we integrate $\sin(\omega t)\sin(\omega t+\phi)$*?*

Use the trigonometric identity (4.59)

$$2\sin(A)\sin(B)=\cos(A-B)-\cos(A+B)$$
$$2\sin(\omega t)\sin(\omega t+\phi)=\cos\left[\omega t-(\omega t+\phi)\right]-\cos\left[\omega t+(\omega t+\phi)\right]$$
$$=\cos(-\phi)-\cos(2\omega t+\phi)$$
$$=\underbrace{\cos(\phi)}_{\text{by (4.51)}}-\cos(2\omega t+\phi)$$

Dividing by 2 we have

$$\sin(\omega t)\sin(\omega t+\phi)=\frac{1}{2}\left[\cos(\phi)-\cos(2\omega t+\phi)\right]$$

Substituting this into (*)

$$P=\frac{\omega VI}{4\pi}\int_0^{2\pi/\omega}\left[\cos(\phi)-\cos(2\omega t+\phi)\right]dt \qquad \left(\text{Taking Out } \frac{1}{2}\right)$$
$$=\frac{\omega VI}{4\pi}\left[\underbrace{t\cos(\phi)}_{\cos(\phi)\text{ is a constant}}-\frac{\sin(2\omega t+\phi)}{2\omega}\right]_0^{2\pi/\omega}$$
$$=\frac{\omega VI}{4\pi}\left(\underbrace{\left(\frac{2\pi\cos(\phi)}{\omega}-\frac{\sin\left\{\left(2\omega\frac{2\pi}{\omega}\right)+\phi\right\}}{2\omega}\right)}_{\text{substituting } t=2\pi/\omega}-\underbrace{\left(0-\frac{\sin(0+\phi)}{2\omega}\right)}_{\text{substituting } t=0}\right)$$

(4.51) $\qquad \cos(-A)=\cos(A)$

$$P=\frac{\omega VI}{4\pi}\left[\frac{2\pi}{\omega}\cos(\phi)-\frac{\sin(4\pi+\phi)}{2\omega}+\frac{\sin(\phi)}{2\omega}\right] \qquad (**)$$

Since the sine function repeats itself after 2π so

$$\sin(4\pi+\phi)=\sin(\phi)$$

Substituting $\sin(4\pi+\phi)=\sin(\phi)$ into (**):

$$P=\frac{\omega VI}{4\pi}\left[\frac{2\pi}{\omega}\cos(\phi)-\frac{\sin(\phi)}{2\omega}+\frac{\sin(\phi)}{2\omega}\right]$$

$$=\frac{\omega VI}{4\pi}\left[\frac{2\pi}{\omega}\cos(\phi)\right]$$

$$P=\frac{VI}{2}\cos(\phi)$$

6. *Which substitution should we use?*

Use (8.55), $u=a\sec(\theta)$. Differentiating

$$\frac{du}{d\theta}\underbrace{=}_{\text{by (6.22)}}a\sec(\theta)\tan(\theta)$$

$$du=a\sec(\theta)\tan(\theta)\,d\theta$$

Substituting $u=a\sec(\theta)$ and $du=a\sec(\theta)\tan(\theta)d\theta$ into the original integral gives:

$$\int\frac{du}{u\sqrt{u^2-a^2}}=\int\frac{a\sec(\theta)\tan(\theta)\,d\theta}{a\sec(\theta)\sqrt{a^2\sec^2(\theta)-a^2}}$$

$$=\int\frac{\tan(\theta)\,d\theta}{\sqrt{a^2\sec^2(\theta)-a^2}} \qquad (*)$$

The denominator, $\sqrt{a^2\sec^2(\theta)-a^2}$ simplifies to

$$\sqrt{a^2\sec^2(\theta)-a^2}=\sqrt{a^2\underbrace{\left(\sec^2(\theta)-1\right)}_{=\tan^2(\theta)}}$$

$$=\sqrt{a^2\tan^2(\theta)}=a\tan(\theta)$$

Substituting this into (*) gives:

$$\int\frac{du}{u\sqrt{u^2-a^2}}=\int\frac{\tan(\theta)}{a\tan(\theta)}d\theta=\frac{1}{a}\int d\theta=\frac{1}{a}\theta+C \qquad (\dagger)$$

We can find θ from our substitution:

$$a\sec(\theta)=u$$

$$\sec(\theta)=\frac{u}{a}$$

Taking inverse $\sec$, $\sec^{-1}$, of both sides: $\theta=\sec^{-1}\left(\frac{u}{a}\right)$

Putting this into $(\dagger)$ displays our result

$$\int\frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)+C$$

(6.22) $\left[\sec(\theta)\right]'=\sec(\theta)\tan(\theta)$

7. *We need to use a substitution, but which substitution?*
By (8.54) we can try $u = a\sin(\theta)$. We need to replace du, so differentiating gives

$$\frac{du}{d\theta} = a\cos(\theta) \text{ which gives } du = a\cos(\theta)\,d\theta$$

Replacing $u = a\sin(\theta)$ and $du = a\cos(\theta)d\theta$ into the original integral gives:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2 - a^2\sin^2(\theta)}} \qquad (*)$$

The denominator $\sqrt{a^2 - a^2\sin^2(\theta)}$ simplifies to

$$\sqrt{a^2 - a^2\sin^2(\theta)} = \sqrt{a^2\left(1 - \sin^2(\theta)\right)}$$
$$= a\sqrt{\underbrace{1 - \sin^2(\theta)}_{=\cos^2(\theta)}} = a\sqrt{\cos^2(\theta)} = a\cos(\theta)$$

Substituting this into (*) gives:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a\cos(\theta)\,d\theta}{a\cos(\theta)}$$
$$= \int d\theta = \theta + C$$

We need to replace θ from the substitution

$$a\sin(\theta) = u$$

We have

$$\sin(\theta) = \frac{u}{a}$$
$$\theta = \sin^{-1}\left(\frac{u}{a}\right)$$

Thus

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$