Complete Solutions to Miscellaneous Exercise 1

1. Cancel out $\frac{1}{2}\rho$ to give $\frac{v^2-u^2}{v^2}=\frac{v^2}{v^2}-\frac{u^2}{v^2}=1-\frac{u^2}{v^2}=1-\left(\frac{u}{v}\right)^2$

2. We have

$$P_2-P_1=\rho g(d-d_2)-\rho g(d-d_1)$$
$$=\rho g[d-d_2-(d-d_1)] \text{(taking out the common factor } \rho g\text{)}$$
$$=\rho g\left[\underbrace{d-d}_{=0}-d_2+d_1\right]$$
$$=\rho g(d_1-d_2)$$
$$=-\rho g(d_2-d_1)$$

3. We have

$$h=\frac{v_2}{g}(v_2-v_1)-\frac{v_2^2-v_1^2}{2g}$$

To obtain a common denominator of 2g, multiply the first term of the Right Hand Side by $\frac{2}{2}(=1)$:

$$h=\frac{2v_2}{2g}(v_2-v_1)-\frac{v_2^2-v_1^2}{2g}$$
$$=\frac{1}{\underbrace{2g}_{\text{taking out the common factor}}}\left[2v_2(v_2-v_1)-(v_2^2-v_1^2)\right]$$
$$=\frac{1}{2g}\left[2v_2^2-2v_2v_1-v_2^2+v_1^2\right]$$
$$=\frac{1}{2g}\left[v_1^2-2v_1v_2+v_2^2\right]$$
$$=\frac{1}{2g}\underbrace{\left[v_1-v_2\right]^2}_{\text{by (1.14)}}$$
$$h=\frac{(v_1-v_2)^2}{2g}$$

4. We have 41, 43, 47, 53, 61 and 71. All answers are prime numbers.

5. (a) Straightforward substitution gives $f_0=2.25\text{ kHz}$.

(b) First transpose the formula to make C the subject:

$$f_0=\frac{1}{2\pi\sqrt{LC}}$$

Taking inverse of both sides gives

$$2\pi\sqrt{LC}=\frac{1}{f_0}$$

(1.14) $$a^2-2ab+b^2=(a-b)^2$$

Dividing both sides by 2π gives

$$\sqrt{LC} = \frac{1}{2\pi f_0}$$

$$LC = \left(\frac{1}{2\pi f_0}\right)^2 = \frac{1}{(2\pi f_0)^2}$$

$$C = \frac{1}{(2\pi f_0)^2 L}$$

To find the value for C substitute $f_0 = 1000$ and $L = 1\times10^{-3}$

$$C = \frac{1}{(2\pi\times 1000)^2 \times (1\times 10^{-3})}$$

$$= 25.3\,\mu\text{F}$$

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6. Simple transform of formula.

7. Substitute the values and add to find $\frac{1}{\text{R}}$ and then take the inverse to obtain $\text{R} = 1000\Omega$.

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8. We can factorize the Right Hand Side to give

$$D = \frac{1}{2}\rho A\left(C_D + kC_L^2\right)v^2$$

$$2D = \rho A\left(C_D + kC_L^2\right)v^2$$

Divide both sides by $\rho A\left(C_D + kC_L^2\right)$;

$$v^2 = \frac{2D}{\rho A\left(C_D + kC_L^2\right)}$$

$$v = \sqrt{\frac{2D}{\rho A\left(C_D + kC_L^2\right)}}$$

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9. Factorize the $\frac{1}{2}\text{I}$ and then multiply both sides by 2 and eventually divide through by $\text{v}_1^2 - \text{v}_2^2$ to give $\text{I} = \frac{2\text{E}}{\text{v}_1^2 - \text{v}_2^2}$.

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10. We have

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + h_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + h_2$$

Subtract $\frac{\text{p}_1}{\rho\text{g}}$ and h_1 from both sides:

$$\frac{v_1^2}{2g} = \frac{p_2}{\rho g} - \frac{p_1}{\rho g} + \left(h_2 - h_1\right) + \frac{v_2^2}{2g}$$

$$= \frac{p_2 - p_1}{\rho g} + \left(h_2 - h_1\right) + \frac{v_2^2}{2g}$$

Multiply both sides by 2g

$$v_1^2 = 2g\frac{(p_2 - p_1)}{\rho g} + 2g(h_2 - h_1) + 2g\frac{v_2^2}{2g}$$

$$= \left[\frac{2(p_2 - p_1)}{\rho}\right] + 2g(h_2 - h_1) + v_2^2$$

So taking the square root gives

$$v_1 = \left[\frac{2(p_2 - p_1)}{\rho} + 2g(h_2 - h_1) + v_2^2\right]^{1/2}$$

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11. (a) $5x = 1$ gives $x = \frac{1}{5}$

(b) $3x + 2 = 8$ leads to $3x = 8 - 2 = 6$, $x = \frac{6}{3} = 2$

(c) $(x-1)(x+2) = 0$ gives

$$x - 1 = 0 \text{ or } x + 2 = 0$$

$$x = 1 \text{ or } x = -2$$

(d) $(3x-1)(2x+3) = 0$ gives

$$3x - 1 = 0 \text{ or } 2x + 3 = 0$$

$$3x = 1 \text{ or } 2x = -3$$

$$x = \frac{1}{3} \text{ or } x = -\frac{3}{2}$$

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12. Substituting $y = 10, u = 14$ into the equation gives

$$10 = 14t - \left(\frac{1}{2} \times 9.8 \times t^2\right)$$

$$= 14t - 4.9t^2$$

Rearranging

$$4.9t^2 - 14t + 10 = 0 \qquad (*)$$

How do we find t?

(*) is a quadratic equation so use formula (1.16) with $a = 4.9$, $b = -14$ and $c = 10$.

$$t = \frac{14 \pm \sqrt{(-14)^2 - (4 \times 4.9 \times 10)}}{2 \times 4.9}$$

$$= \frac{14 \pm 0}{9.8}$$

$$t = 1.43s$$

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13. Squaring both sides of u gives

$$u^2 = \frac{2\gamma P_1 V_1}{\gamma - 1}\left[1 - \frac{P_2}{P_1} \cdot \frac{V_2}{V_1}\right] \qquad (*)$$

From the question we have

(1.16) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$P_2V_2^{\gamma} = P_1V_1^{\gamma}$$

$$\frac{V_2^{\gamma}}{V_1^{\gamma}} = \frac{P_1}{P_2}$$

$$\left(\frac{V_2}{V_1}\right)^{\gamma} = \frac{P_1}{P_2}$$

$$\frac{V_2}{V_1} = \left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}}$$

Substituting $\frac{V_2}{V_1} = \left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}}$ into $1-\frac{P_2}{P_1}\cdot\frac{V_2}{V_1}$ gives

$$1-\frac{P_2}{P_1}\cdot\frac{V_2}{V_1} = 1-\frac{P_2}{P_1}\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}}$$

$$= 1-\frac{P_2}{P_1}\left(\frac{P_2}{P_1}\right)^{-\frac{1}{\gamma}}$$

$$= 1-\underbrace{\left(\frac{P_2}{P_1}\right)^{1-\frac{1}{\gamma}}}_{\text{by (1.5)}}$$

Putting this into (*) gives

$$u^2 = \frac{2\gamma P_1V_1}{\gamma-1}\left[1-\left(\frac{P_2}{P_1}\right)^{1-\frac{1}{\gamma}}\right] \qquad (\dagger)$$

To evaluate u we need to find P_2 first, how?
Use

$$P_2V_2^{\gamma} = P_1V_1^{\gamma}$$

$$P_2 = P_1\left(\frac{V_1}{V_2}\right)^{\gamma}$$

Substitute

$$\gamma = 1.39,\ P_1 = 5.2\times10^6,\ V_1 = 3.1\times10^{-3},\ V_2 = 5\times10^{-3}$$

into the above

$$P_2 = (5.2\times10^6)\left(\frac{3.1\times10^{-3}}{5\times10^{-3}}\right)^{1.39} = 2.68\times10^6$$

and $P_2 = 2.68\times10^6$ into $(\dagger)$ gives

(1.5) $$a^m a^n = a^{m+n}$$

$$u^2 = \frac{\left(2\times1.39\times5.2\times10^6\times3.1\times10^{-3}\right)}{1.39-1}\left[1-\left(\frac{2.68\times10^6}{5.2\times10^6}\right)^{1-\frac{1}{1.39}}\right]$$

$$=114906.67\left[1-\left(\frac{2.68}{5.2}\right)^{0.28}\right]$$

$$=19464.12$$

$$u = 140 \text{ m/s}$$

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14. By transposing and expanding you result in a quadratic $R^2+8R-39=0$. Use (1.16) with $a=1,\ b=8$ and $c=-3$ to give $R=3.42\Omega$.

15.(a)

$$x^2-7x+10=(x-5)(x-2)=0$$

$$x-5=0 \text{ or } x-2=0$$

$$x=5 \text{ or } x=2$$

(b) By (1.15)

$$x^2-1=(x-1)(x+1)=0 \quad \text{which gives } x=1,\ x=-1.$$

(c) $2x^2-3x+1=(2x-1)(x-1)=0$. So we have

$$2x-1=0,\ x-1=0$$

$$x=\frac{1}{2},\ x=1$$

(d) $15x^2-x-2=(5x-2)(3x+1)=0$. This gives

$$5x-2=0,\ 3x+1=0$$

$$5x=2,\ 3x=-1$$

$$x=\frac{2}{5},\ x=-\frac{1}{3}$$

(e) First divide through by -100, so we have

$$x^2-4x+3=0$$

$$(x-3)(x-1)=0$$

$$x=3,\ x=1$$

16. Use TABLE 1. You should get all coefficients into $M^0L^0T^0$ - dimensionless.

17. Factorize by taking out the common factor of $\frac{w}{36EI}$:

$$y=\frac{w}{36EI}(x^4-4Lx^3+L^4)$$

18. (i) Straightforward transposition gives $P=\frac{n^2\pi^2EI}{L^2}$ (ii) $P=12\times10^6$ N

(1.15) $$a^2-b^2=(a-b)(a+b)$$

(1.16) $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

19. (a) Put $a = 1$, $b = 3$ and $c = 1$ into (1.16) to solve $x^2 + 3x + 1 = 0$:

$$x = \frac{-3 \pm \sqrt{9 - (4 \times 1 \times 1)}}{2}$$
$$= \frac{-3 \pm \sqrt{5}}{2}$$
$$x = \frac{-3 + \sqrt{5}}{2} \text{ or } x = \frac{-3 - \sqrt{5}}{2}$$
$$x = -0.38 \text{ or } x = -2.62$$

(b) Similarly put $a = 1$, $b = 4$ and $c = 2$ into (1.16), hence

$$x = -0.59\,,\; x = -3.41$$

(c) With $a = 5$, $b = 2$ and $c = -1$ into (1.16) gives $x = 0.29$, $x = -0.69$.

(d) With $a = -2$, $b = -3$ and $c = 1$ into (1.16) gives $x = 0.28$, $x = -1.78$.

20. We have

$$C = \frac{F_0}{k - m\alpha^2}$$
$$\underbrace{=}_{\substack{\text{divide numerator}\\ \text{and denominator by } k}} \frac{F_0/k}{1 - \frac{m\alpha^2}{k}}$$
$$= \frac{F_0/k}{1 - \frac{\alpha^2}{k/m}}$$
$$= \frac{F_0/k}{1 - \frac{\alpha^2}{\omega^2}} \quad \text{because} \quad \omega = \sqrt{k/m}$$
$$= \frac{F_0/k}{1 - r^2} \quad \text{because} \quad r = \frac{\alpha}{\omega}$$

21. Expand the brackets and solve the simultaneous equations to give $I_1 = 0.022$ A and $I_2 = 0.07$ A.

22. Equating $P = \rho RT$ and $P = k\rho^\gamma$ gives

$$\rho RT = k\rho^\gamma$$
$$\frac{\rho^\gamma}{\rho} = \frac{RT}{k} \quad \text{which gives} \quad \underbrace{\rho^{\gamma-1}}_{\text{by (1.6)}} = \frac{RT}{k}$$

Substituting $\rho^{\gamma-1} = \frac{RT}{k}$ into $v = (\gamma k\rho^{\gamma-1})^{\frac{1}{2}}$ gives

$$v = \left[\gamma k\left(\frac{RT}{k}\right)\right]^{\frac{1}{2}}$$
$$= (\gamma RT)^{\frac{1}{2}}$$
$$v = \sqrt{\gamma RT}$$

(1.6) $\frac{a^m}{a^n} = a^{m-n}$ (1.16) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

23. Taking out the Q gives

$$d_1^2 - d_2^2 = \frac{2Q^2}{g}\left(\frac{1}{d_2} - \frac{1}{d_1}\right)$$

$$\underbrace{\left(d_1 - d_2\right)\left(d_1 + d_2\right)}_{\text{by (1.15)}} = \frac{2Q^2}{g}\left(\frac{d_1 - d_2}{d_1 d_2}\right)$$

Divide both sides by $(d_1 - d_2)$:

$$d_1 + d_2 = \frac{2Q^2}{g d_1 d_2} \qquad (*)$$

We are also given

$$F = \frac{Q}{\sqrt{g d_1^3}}$$

$$Q = F\sqrt{g d_1^3}$$

Squaring both sides gives

$$Q^2 = F^2 g d_1^3$$

Substituting this into (*) gives

$$d_1 + d_2 = \frac{2F^2 g d_1^3}{g d_1 d_2}$$

$$= \frac{2F^2 d_1^2}{d_2}$$

Multiplying both sides by d_2:

$$d_2^2 + d_2 d_1 = 2F^2 d_1^2$$

Dividing both sides by d_1^2 gives

$$\frac{d_2^2 + d_2 d_1}{d_1^2} = 2F^2$$

$$\frac{d_2^2}{d_1^2} + \frac{d_2 d_1}{d_1^{\ 2}} = 2F^2$$

$$\left(\frac{d_2}{d_1}\right)^2 + \frac{d_2}{d_1} - 2F^2 = 0 \qquad (**)$$

Putting $x = \dfrac{d_2}{d_1}$ into (**) we have a quadratic equation $x^2 + x - 2F^2 = 0$

How do we solve for x?

Use formula (1.16) with $a = 1$, $b = 1$ and $c = -2F^2$

$$x = \frac{-1 \pm \sqrt{1 - \left[4 \times \left(-2F^2\right)\right]}}{2}$$

$$= \frac{-1 \pm \sqrt{1 + 8F^2}}{2}$$

(1.15) $a^2 - b^2 = (a - b)(a + b)$ (1.16) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Putting $x = \frac{d_2}{d_1}$ as before we obtain our result

$$\frac{d_2}{d_1} = \frac{1}{2}\left[-1 \pm \sqrt{1 + 8F^2}\right]$$

24. (a) $x = 1$, $y = 1$ (b) $x = 58/5$, $y = -33/5$ (c) $x = 1.77$, $y = 0.85$

25. From $\omega^2 = \frac{JG}{IL}$ we have

$$L = \frac{JG}{\omega^2 I}$$

Substituting this into $\alpha^2 = \frac{JG}{(I + 2mr^2)L}$ gives

$$\alpha^2 = \frac{JG}{(I + 2mr^2)\frac{JG}{\omega^2 I}}$$

$$= \frac{\omega^2 I}{I + 2mr^2}$$

Multiplying both sides by $I + 2mr^2$ gives

$$(I + 2mr^2)\alpha^2 = \omega^2 I$$

$$I\alpha^2 + 2mr^2\alpha^2 = \omega^2 I$$

$$\omega^2 I - \alpha^2 I = 2mr^2\alpha^2$$

$$I(\omega^2 - \alpha^2) = 2mr^2\alpha^2$$

$$I = \frac{2mr^2\alpha^2}{\omega^2 - \alpha^2}$$

26. Use formula (1.16) with $a = m,\ b = \zeta$ and $c = -3$. For the second part we have

$$1 - \frac{4mk}{\zeta^2} = 0$$

$$\zeta^2 = 4mk$$

$$\zeta = \sqrt{4mk}$$

$$= 2\sqrt{mk}$$

Ignore the negative root because $\zeta > 0$ in question.

27. Apply transposition.

28. Put $x = \omega^2$ into the given equation to get

$$x^2 - 402x + 800 = 0 \text{ (remember } x^2 = (\omega^2)^2 = \omega^4\text{)}$$

$$(x - 400)(x - 2) = 0$$

$$x = 400,\ x = 2$$

Putting back $\omega^2 = x$ we get

$$\omega^2 = 400,\ \omega^2 = 2$$

Thus

$$\omega_1 = 20,\ \omega_2 = -20,\ \omega_3 = \sqrt{2} \text{ and } \omega_4 = -\sqrt{2}$$

Remember the square root gives a positive and negative root.

29.Putting $y=\omega^2$ into the given equation

$$\omega^4-2\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)\omega^2+\left(\frac{g^2}{\ell^2}+\frac{2kx^2g}{m\ell^3}\right)=0$$

gives

$$y^2-2\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)y+\left(\left(\frac{g}{\ell}\right)^2+\frac{2kx^2g}{m\ell^3}\right)=0$$

Solve for y by using (1.16) with $a=1,\ b=-2\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)$ and $c=\left(\frac{g}{\ell}\right)^2+\frac{2kx^2g}{m\ell^3}$:

$$y=\frac{2\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)\pm\sqrt{4\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)^2-4\left[\left(\frac{g}{\ell}\right)^2+\frac{2kx^2g}{m\ell^3}\right]}}{2}$$

$$=\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)\pm\sqrt{\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)^2-\left(\frac{g}{\ell}\right)^2-\frac{2kx^2g}{m\ell^3}} \qquad (\dagger)$$

We can simplify the expression under the square root:

$$\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)^2-\left(\frac{g}{\ell}\right)^2-\frac{2kx^2g}{m\ell^3}$$

$$=\underbrace{\left(\frac{g}{\ell}\right)^2+2\frac{g}{\ell}\frac{kx^2}{m\ell^2}+\left(\frac{kx^2}{m\ell^2}\right)^2}_{\text{by (1.13)}}-\left(\frac{g}{\ell}\right)^2-\frac{2kx^2g}{m\ell^3}$$

$$=\left(\frac{kx^2}{m\ell^2}\right)^2 \qquad \text{(the other terms cancel out).}$$

Substituting this into $(\dagger)$ gives

$$y=\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)\pm\sqrt{\left(\frac{kx^2}{m\ell^2}\right)^2}$$

$$=\left(\frac{g}{\ell}+\frac{kx^2}{m\ell^2}\right)\pm\frac{kx^2}{m\ell^2}$$

$$y=\frac{g}{\ell}+\underbrace{\frac{kx^2}{m\ell^2}+\frac{kx^2}{m\ell^2}}_{=\frac{2kx^2}{m\ell^2}} \text{ or } y=\frac{g}{\ell}+\underbrace{\frac{kx^2}{m\ell^2}-\frac{kx^2}{m\ell^2}}_{=0}$$

So $y=\frac{g}{\ell}+\frac{2kx^2}{m\ell^2}$ or $y=\frac{g}{\ell}$

Remember $y=\omega^2$; So

$$\omega=\pm\sqrt{\frac{g}{\ell}+\frac{2kx^2}{m\ell^2}},\ \omega=\pm\sqrt{\frac{g}{\ell}}$$

(1.13) $(a+b)^2=a^2+2ab+b^2$ (1.16) $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\omega_1 = \sqrt{\frac{g}{\ell}}\ ,\ \omega_2 = -\omega_1,\ \ \omega_3 = \sqrt{\frac{g}{\ell} + \frac{2kx^2}{m\ell^2}}\ \text{and}\ \ \omega_4 = -\omega_3$$

30. Make L the subject of $kL^2 = Z$, substitute into w to obtain formula.

31. Taking out the common factor of $\frac{w}{6}$ on the Right Hand Side gives

$$EIy = \frac{w}{6}\left[(L-x)L - \frac{(L-x)^3}{L}\right]$$

Dividing both sides by EI gives

$$y = \frac{w}{6EI}\left[(L-x)L - \frac{(L-x)^3}{L}\right]$$

By question we know $w \neq 0$ and $EI \neq 0$ so the expression in the square brackets must equal zero to give zero deflection, $y = 0$. Thus

$$(L-x)L - \frac{(L-x)^3}{L} = 0$$

What do you notice about Left Hand Side?

$(L-x)$ is common to both terms because $(L-x)^3 \underbrace{=}_{\text{by (1.5)}} (L-x)(L-x)^2$

So we take out the common factor

$$(L-x)\left[L - \frac{(L-x)^2}{L}\right] = 0$$

We know have $(L-x) = 0$ or $\left[L - \frac{(L-x)^2}{L}\right] = 0$. The first equation gives $x = L$.

How can we find x in the second equation $\left[L - \frac{(L-x)^2}{L}\right] = 0$?

Multiplying both sides by L gives

$$L^2 - (L-x)^2 = 0$$

Expanding the second term:

$$L^2 - \underbrace{\left(L^2 - 2Lx + x^2\right)}_{\text{by (1.14)}} = 0$$

$$\underbrace{L^2 - L^2}_{=0} + 2Lx - x^2 = 0$$

Taking out the common factor of x:

$$x(2L - x) = 0$$

$$x = 0 \text{ or } x = 2L$$

The deflection is zero at $x = 0$ and $x = L$. Also x ***cannot equal*** $2L$ because the length of the beam is L.

(1.5) $a^m a^n = a^{m+n}$

(1.14) $(a-b)^2 = a^2 - 2ab + b^2$