Complete solutions to Exercise 10(a)

1\. (a) $u+v=(2+j3)+(5+j8)=(2+5)+j(3+8)=7+j11$

(b) $u-v=(2+j3)-(5+j8)=(2-5)+j(3-8)=-3-j5$

(c)Use FOIL.

$$uv=(2+j3)(5+j8)=(2\times5)+j(8\times2)+j(3\times5)+j^2(3\times8)$$
$$=10+j16+j15+j^2 24$$
$$=10+j31-24 \qquad [\text{Because } j^2=-1]$$
$$=-14+j31$$

(d) $vu=(5+j8)(2+j3)=-14+j31$, same as uv.

(e) $\frac{u}{v}=\frac{2+j3}{5+j8}$. What is the complex conjugate of $5+j8$?

$5-j8$. Multiplying numerator and denominator by $5-j8$

$$\frac{2+j3}{5+j8}=\frac{(2+j3)(5-j8)}{5^2+8^2}=\frac{10-j16+j15-j^2 24}{89}$$
$$=\frac{10-j+24}{89}=\frac{34-j}{89}=\frac{34}{89}-j\frac{1}{89}=0.382-j0.011$$

(f) Similarly $v/u=2.615+j0.077$

Also (g) $9+j14$ (h) $-5+j12$ (i) $-39+j80$ (j) $-44+j92$

2\. $3+j15,\ 3-j15$. The complex conjugate of $j3=0+j3$ is $0-j3=-j3$.

The complex conjugate of j is $-j$.

The complex conjugate of $0=0+j0$ is $0-0j=0$.

The complex conjugate of π is π.

The complex conjugate of e is e.

3\. First we evaluate z^2

$$z^2=(3-j)^2=(3-j)(3-j)=8-j6$$

Substituting $z^2=8-j6,\ z=3-j$ into $z^2+7z+13$ gives

1 $$z^2+7z+13=8-j6+7(3-j)+13$$
$$=8-j6+21-j7+13=42-j13$$

4\. By (10.2) $j^{20}=(j^4)^5=1^5=1$

By (10.3) $j^9=j^8 j=(j^4)^2 j=1^2 j=j$

By (10.3) $j^{23}=j^{20}j^3=1j^3=1\underbrace{(-j)}_{\text{by (10.9)}}=-j$

By (10.2) $j^{100}=(j^4)^{25}=1$

By (10.3) $j^{1007}=j^{1000}.j^7=1.j^7=j^4 j^3=1j^3=-j \quad [\text{By (10.9)}]$

(10.2) $(a^m)^n=a^{m\times n}$

(10.3) $a^m a^n=a^{m+n}$

(10.9) $j^3=-j$

5. (i) We have $z = 2 + j$

$$z^2 = (2+j)(2+j) = 4 + j2 + j2 + j^2 = 4 + j4 - 1 = 3 + j4$$

$$z^3 = z^2 z = (3+j4)(2+j) = 2 + j11$$

Similarly $z^4 = z^3 z = (2+j11)(2+j) = -7 + j24$. Substituting all this into $z^4 - 6z^2 + 25$ gives

$$z^4 - 6z^2 + 25 = -7 + j24 - 6(3+j4) + 25 = 0$$

(ii) Straightforward, $z = 2 + j$, as shown in part(i), is a root.

---

6. We have

$$Y = \frac{1}{75 - j40} \underbrace{=}_{(10.13)} \frac{75 + j40}{75^2 + 40^2} = \frac{75 + j40}{7225} = (0.0104 + j0.0055)\text{S}$$

(S=siemens).

7.(a) Equating real and imaginary parts gives $a = 2,\ b = 3$

(b) $(2 + j3) - (2 - j3) = 0 + j6$ which gives $a = 0,\ b = 6$

(c) $(1 + j)^2 = (1 + j)(1 + j) = 2j$ which gives $a = 0,\ b = 2$

(d) $\dfrac{1}{3 - j4} \underbrace{=}_{\text{by (10.13)}} \dfrac{3 + j4}{3^2 + 4^2} = \dfrac{3}{25} + j\dfrac{4}{25} = 0.12 + j0.16$ which gives $a = 0.12,\ b = 0.16$

(e) $(1 + j2)^2 = (1 + j2)(1 + j2) = 1 + j2 + j2 + j^2 4 = -3 + j4$

$2a + jb = -3 + j4$ gives $2a = -3,\ b = 4$ and so $a = -\dfrac{3}{2},\ b = 4$

(f)

$$\frac{7 + j5}{2 - j} \underbrace{=}_{\text{by (10.13)}} \frac{(7 + j5)(2 + j)}{2^2 + 1^2} = \frac{14 + j7 + j10 + j^2 5}{5}$$

$$= \frac{9 + j17}{5}$$

$$= 1.8 + j3.4$$

So $1.8a + j3.4b = 1.8 + j3.4$, equating real and imaginary parts gives $a = 1,\ b = 1$.

---

2 8. (a) $-j5(3 + j4) = -j15 - j^2 20 = 20 - j15$ [Remember $j^2 = -1$]

(b) $\dfrac{3 + j4}{j5}$. Need to find the complex conjugate of $j5 = 0 + j5$ which is $0 - j5 = -j5$.

$$\frac{3 + j4}{j5} = \frac{-j5(3 + j4)}{5^2} = \frac{\overbrace{20 - j15}^{\text{by part (a)}}}{25} = 0.8 - j0.6$$

(c) $(2 - j3)(1 + j) = 2 + j2 - j3 - j^2 3 = 2 - j + 3 = 5 - j$

(d)

$$\frac{3 + j4}{j5} - \frac{2 - j3}{1 - j} = \underbrace{0.8 - j0.6}_{\text{by part (b)}} - \frac{(2 - j3)(1 + j)}{1^2 + 1^2} = 0.8 - j0.6 - \frac{\overbrace{5 - j}^{\text{by part (c)}}}{2} = -1.7 - j0.1$$

(10.13) $$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2}$$

9. We have $\frac{1}{Z_t} = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_2 + Z_1}{Z_1 Z_2}$. So $Z_t = \frac{Z_1 Z_2}{Z_2 + Z_1}$ and substituting $Z_1 = 1 + j,\ Z_2 = 3 + j4$ into this gives

$$Z_t = \frac{(1+j)(3+j4)}{1+j+3+j4} = \frac{3+j4+j3+j^2 4}{4+j5} = \frac{-1+j7}{4+j5}$$

How can we write Z_t in $a + jb$ form?

Find the complex conjugate of $4 + j5$ and then use (10.13).

Complex conjugate of $4 + j5$ is $4 - j5$ so

$$Z_t \underbrace{=}_{\text{by (10.13)}} \frac{(-1+j7)(4-j5)}{4^2+5^2} = \frac{31+j33}{41}$$

$$Z_t = (0.756 + j0.805)\Omega$$

---

10. By taking out the common factor $20I$ we have

$$20I(1+j5) = 200$$

$$I = \frac{200}{20(1+j5)} = \frac{10}{1+j5}$$

$$\underbrace{=}_{by\ \text{(10.13)}} \frac{10(1-j5)}{1^2+5^2} = \frac{10-j50}{26}$$

$$I = (0.385 - j1.923\ )\ \text{A}$$

11. For each of these equations we need to use the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(a) For $z^2 + 2z + 26 = 0$ we have $a = 1,\ b = 2$ and $c = 26$ which gives

$$z = \frac{-2 \pm \sqrt{4 - (4 \times 1 \times 26)}}{2} = \frac{-2 \pm \sqrt{-100}}{2}$$

$$= \frac{-2}{2} \pm \frac{\sqrt{-100}}{2}$$

3

$$= -1 \pm \frac{j10}{2} = -1 \pm j5$$

(b) $z^2 - 2z + 3 = 0$. By applying the quadratic formula we have

$$z = \frac{2 \pm \sqrt{4 - (4 \times 3)}}{2} = \frac{2 \pm \sqrt{-8}}{2}$$

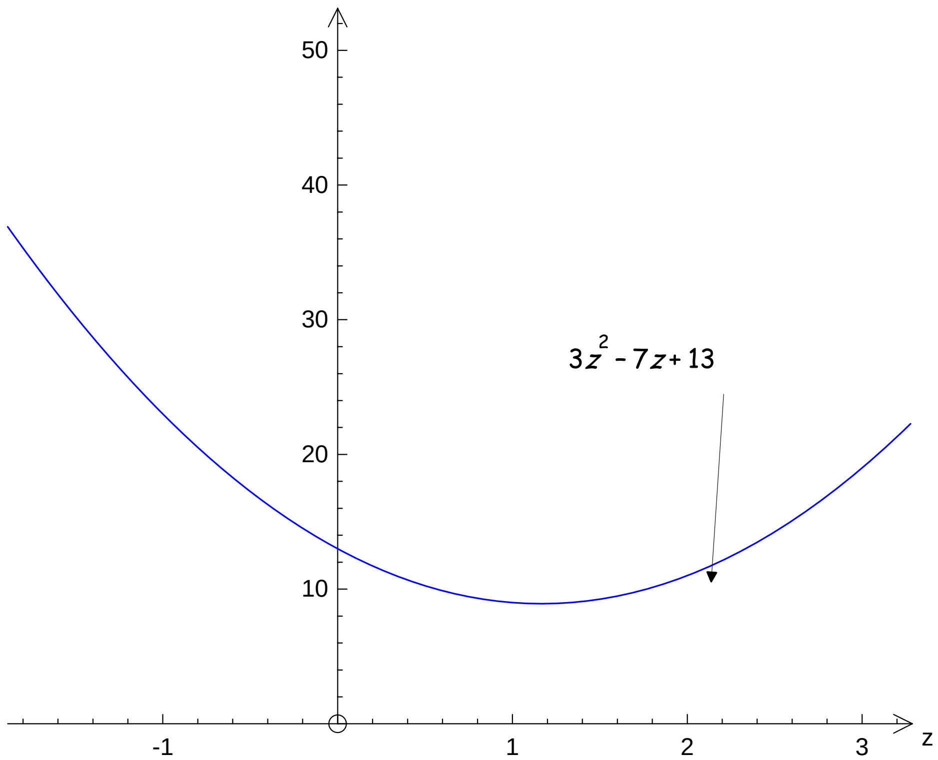
$$= 1 \pm \frac{j\sqrt{8}}{2}$$

$$= 1 \pm \frac{j\overbrace{\sqrt{4}\sqrt{2}}^{\text{by (10.1)}}}{2} = 1 \pm j\sqrt{2}$$

(10.1) $$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

(10.13) $$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{c^2+d^2}$$

(c) $3z^2 - 7z + 13 = 0$. Similarly $z = \frac{7}{6} \pm j\frac{\sqrt{107}}{6}$.



From the graphs it is clear that the curves do not touch the *z*-axis and therefore there are no real roots

12. $I = (3 + j6)/(8 + j) = (0.462 + j0.692)A$

13. Let

$$10 = 4I_p + j6I_p - j4I_s \qquad (\dagger)$$

$$0 = 50I_s + j12I_s + 8I_s - j4I_p \qquad (\dagger\dagger)$$

Making $j4I_p$ the subject of $(\dagger\dagger)$

4 $$j4I_p = (50 + j12 + 8)I_s = (58 + j12)I_s$$

So

$$I_p = \left[\frac{58 + j12}{j4}\right]I_s$$

The complex conjugate of $j4$ is $-j4$ so multiplying numerator and denominator by $-j4$ gives

$$I_p = \left[\frac{(58 + j12)(-j4)}{4^2}\right]I_s$$

$$= \left[\frac{-j232 - j^2 48}{16}\right]I_s$$

$$= \left[\frac{48 - j232}{16}\right]I_s$$

$$I_p = [3 - j14.5]I_s \qquad (*)$$

Rewrite (†) as

$$10 = (4 + j6) I_p - j4 I_s$$

Substituting for I_p from (*) gives

$$\begin{aligned} 10 &= (4 + j6)(3 - j14.5) I_s - j4 I_s \\ &= \left[12 - j58 + j18 - j^2 87 - j4\right] I_s \\ &= \left[12 - j40 + 87 - j4\right] I_s \\ 10 &= [99 - j44] I_s \end{aligned}$$

Hence

$$I_s = \frac{10}{99 - j44} \underbrace{=}_{\text{by (10.13)}} \frac{10(99 + j44)}{99^2 + 44^2} = (0.0843 + j0.0375)\text{ A}$$

Substituting this into (*) gives

$$I_p = (3 - j14.5)(0.0843 + j0.0375) = (0.797 - j1.110)\text{ A}$$

---

14. (i) We have

$$X_c = \frac{1}{j\omega C} \quad \text{which gives} \quad \frac{1}{X_c} = j\omega C$$

$$\frac{1}{Z} = \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R}$$

Hence $Z = \dfrac{R}{1 + j\omega CR}$.

(ii) To find the real and imaginary parts of Z we multiply numerator and denominator by the complex conjugate of $1 + j\omega CR$ which is $1 - j\omega CR$:

$$Z = \frac{R(1 - j\omega CR)}{(1 + j\omega CR)(1 - j\omega CR)} = \frac{R - j\omega CR^2}{1^2 + \omega^2 C^2 R^2}$$

So the real part of Z, that is the j terms are zero, is

$$\frac{R}{1^2 + \omega^2 C^2 R^2} = \frac{R}{1 + \omega^2 C^2 R^2}$$

5 The imaginary part of Z, that is only the j terms, is

$$\frac{-\omega CR^2}{1 + \omega^2 C^2 R^2}$$

(10.13) $$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2}$$