Complete solutions to Exercise 11(c)

1. (a) Using (11.8)

$$\det\begin{pmatrix}1 & 3 & -1\\ 2 & 0 & 5\\ -6 & 3 & 1\end{pmatrix} = 1\det\begin{pmatrix}0 & 5\\ 3 & 1\end{pmatrix} - 3\det\begin{pmatrix}2 & 5\\ -6 & 1\end{pmatrix} - 1\det\begin{pmatrix}2 & 0\\ -6 & 3\end{pmatrix}$$

$$\underbrace{=}_{\text{by (11.1)}} 1\left[(0\times 1)-(3\times 5)\right]-3\left[(2\times 1)-(-6\times 5)\right]-1\left[(2\times 3)-(-6\times 0)\right]=-117$$

(b) Similarly

$$\det\begin{pmatrix}2 & -10 & 11\\ 5 & 3 & -4\\ 7 & 9 & 12\end{pmatrix} = 2\det\begin{pmatrix}3 & -4\\ 9 & 12\end{pmatrix} + 10\det\begin{pmatrix}5 & -4\\ 7 & 12\end{pmatrix} + 11\det\begin{pmatrix}5 & 3\\ 7 & 9\end{pmatrix}$$

$$\underbrace{=}_{\text{by (11.1)}} 2\left[(3\times 12)-(9\times(-4))\right]+10\left[(5\times 12)-(7\times(-4))\right]+11\left[(5\times 9)-(7\times 3)\right]$$

$$=1288$$

(c) Very similar to parts (a) and (b). Thus $\det\mathbf{C}=-114$.

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2. Remember for the transpose we interchange the rows and columns of the matrix:

(a) $\begin{pmatrix}-2 & 3\\ 1 & 5\end{pmatrix}^T = \begin{pmatrix}-2 & 1\\ 3 & 5\end{pmatrix}$ (b) $\begin{pmatrix}1/3 & -2/5\\ -3/7 & \pi\end{pmatrix}^T = \begin{pmatrix}1/3 & -3/7\\ -2/5 & \pi\end{pmatrix}$

(c) $\begin{pmatrix}7 & 3 & 4\\ 2 & 6 & 1\\ -3 & -3 & 1\end{pmatrix}^T = \begin{pmatrix}7 & 2 & -3\\ 3 & 6 & -3\\ 4 & 1 & 1\end{pmatrix}$ (d) $\begin{pmatrix}1.17 & 1.36\\ 9.39 & -1.45\\ 2.11 & 5.20\end{pmatrix}^T = \begin{pmatrix}1.17 & 9.39 & 2.11\\ 1.36 & -1.45 & 5.20\end{pmatrix}$

(e) $\begin{pmatrix}-7 & 2 & 1 & 5\\ 3 & 6 & -4 & 7\\ 8 & 3 & -3 & 5\\ -4 & 6 & 7 & 0\end{pmatrix}^T = \begin{pmatrix}-7 & 3 & 8 & -4\\ 2 & 6 & 3 & 6\\ 1 & -4 & -3 & 7\\ 5 & 7 & 5 & 0\end{pmatrix}$ (f) $\begin{pmatrix}1 & -3 & 7\\ -9 & 4 & 6\\ -4 & 2 & 8\\ 9 & 19 & 11\end{pmatrix}^T = \begin{pmatrix}1 & -9 & -4 & 9\\ -3 & 4 & 2 & 19\\ 7 & 6 & 8 & 11\end{pmatrix}$

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3. Using (11.8)

$$\det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 7 & 3 & -2\\ 4 & 2 & 7\end{pmatrix} = \mathbf{i}\left[\det\begin{pmatrix}3 & -2\\ 2 & 7\end{pmatrix}\right] - \mathbf{j}\left[\det\begin{pmatrix}7 & -2\\ 4 & 7\end{pmatrix}\right] + \mathbf{k}\left[\det\begin{pmatrix}7 & 3\\ 4 & 2\end{pmatrix}\right]$$

$$\underbrace{=}_{\text{by (11.1)}} \mathbf{i}\left[(3\times 7)-(2\times(-2))\right]-\mathbf{j}\left[(7\times 7)-(4\times(-2))\right]+\mathbf{k}\left[(7\times 2)-(4\times 3)\right]$$

$$=25\mathbf{i}-57\mathbf{j}+2\mathbf{k}$$

(11.1) $$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}=ad-cb$$

(11.8) $$\det\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = a\left[\det\begin{pmatrix}e & f\\ h & i\end{pmatrix}\right] - b\left[\det\begin{pmatrix}d & f\\ g & i\end{pmatrix}\right] + c\left[\det\begin{pmatrix}d & e\\ g & h\end{pmatrix}\right]$$

4. We have

$$\det\begin{pmatrix}1 & 0 & 3\\ 5 & t & -7\\ 3 & 9 & t-1\end{pmatrix} = 1\det\begin{pmatrix}t & -7\\ 9 & t-1\end{pmatrix} - 0 + 3\det\begin{pmatrix}5 & t\\ 3 & 9\end{pmatrix}$$

$$\underbrace{=}_{\text{by (11.1)}} \left[t(t-1)-(9\times(-7))\right]+3\left[(5\times 9)-(3\times t)\right]$$

$$=\left[t^2-t+63\right]+3\left[45-3t\right]$$

$$=t^2-t+63+135-9t=t^2-10t+198$$

Since the determinant is zero we have

$$t^2-10t+198=0$$

How do we solve this quadratic equation?
Use (1.16) with $a=1$, $b=-10$ and $c=198$

$$t=\frac{10\pm\sqrt{(-10)^2-(4\times1\times198)}}{2}$$

$$=\frac{10\pm\sqrt{-692}}{2}=\frac{10\pm j\sqrt{692}}{2}=5\pm j\sqrt{173}$$

Thus $t=5+j\sqrt{173},\ 5-j\sqrt{173}$

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5. We need to find the cofactors of each element of the matrix.
Cofactor of 1 is

$$\det\begin{pmatrix}3 & 7\\ -1 & 0\end{pmatrix}=(3\times0)-(-1\times7)=7$$

Cofactor of 0 is

$$-\det\begin{pmatrix}-2 & 7\\ 6 & 0\end{pmatrix}=-\left[(-2\times0)-(6\times7)\right]=42$$

Cofactor of 5 is

$$\det\begin{pmatrix}-2 & 3\\ 6 & -1\end{pmatrix}=\left[(-2\times(-1))-(6\times3)\right]=-16$$

Cofactor of -2 is

$$-\det\begin{pmatrix}0 & 5\\ -1 & 0\end{pmatrix}=-\left[(0\times0)-(-1\times5)\right]=-5$$

Cofactor of 3 is

$$\det\begin{pmatrix}1 & 5\\ 6 & 0\end{pmatrix}=\left[(1\times0)-(6\times5)\right]=-30$$

Cofactor of 7 is

$$-\det\begin{pmatrix}1 & 0\\ 6 & -1\end{pmatrix}=-\left[(1\times(-1))-(6\times0)\right]=1$$

(1.16) $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(11.1) $$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}=ad-cb$$

Cofactor of 6 is

$$\det\begin{pmatrix}0 & 5\\ 3 & 7\end{pmatrix}=\left[(0\times 7)-(3\times 5)\right]=-15$$

Cofactor of -1 is

$$-\det\begin{pmatrix}1 & 5\\ -2 & 7\end{pmatrix}=-\left[(1\times 7)-(-2\times 5)\right]=-17$$

Cofactor of 0 is

$$\det\begin{pmatrix}1 & 0\\ -2 & 3\end{pmatrix}=\left[(1\times 3)-(-2\times 0)\right]=3$$

Collecting the cofactors gives the cofactor matrix:

$$\mathbf{C}=\begin{pmatrix}7 & 42 & -16\\ -5 & -30 & 1\\ -15 & -17 & 3\end{pmatrix}$$

Transposing this matrix (interchanging rows and columns) gives

$$\mathbf{C}^T=\begin{pmatrix}7 & -5 & -15\\ 42 & -30 & -17\\ -16 & 1 & 3\end{pmatrix}$$

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6. Writing the equations in matrix form yields

$$\begin{pmatrix}3 & -5 & 3\\ 2 & 1 & -7\\ -10 & 4 & 5\end{pmatrix}\begin{pmatrix}i_1\\ i_2\\ i_3\end{pmatrix}=\begin{pmatrix}7.5\\ -17.5\\ 16\end{pmatrix}$$

Let $\mathbf{A}=\begin{pmatrix}3 & -5 & 3\\ 2 & 1 & -7\\ -10 & 4 & 5\end{pmatrix}$. *What do we need to find?*

The inverse matrix, $\mathbf{A}^{-1}$. The currents can be obtained from

$$\begin{pmatrix}i_1\\ i_2\\ i_3\end{pmatrix}=\mathbf{A}^{-1}\begin{pmatrix}7.5\\ -17.5\\ 16\end{pmatrix}\qquad (*)$$

To find $\mathbf{A}^{-1}$ we have to evaluate the determinant and the adjoint of $\mathbf{A}$.

$$\det\begin{pmatrix}3 & -5 & 3\\ 2 & 1 & -7\\ -10 & 4 & 5\end{pmatrix}=3\det\begin{pmatrix}1 & -7\\ 4 & 5\end{pmatrix}-(-5)\det\begin{pmatrix}2 & -7\\ -10 & 5\end{pmatrix}+3\det\begin{pmatrix}2 & 1\\ -10 & 4\end{pmatrix}$$

$$=3\left[(1\times 5)-(4\times(-7))\right]+5\left[(2\times 5)-(10\times 7)\right]+3\left[(2\times 4)-(-10\times 1)\right]$$

$$=-147$$

Next we find $adj\mathbf{A}$, which is the cofactor matrix transposed. The cofactor matrix can be obtained using the method described in solution 5. Thus

$$\mathbf{C}=\begin{pmatrix}33 & 60 & 18\\ 37 & 45 & 38\\ 32 & 27 & 13\end{pmatrix}$$

Transposing this gives $adj\mathbf{A}$

$$adj\mathbf{A}=\mathbf{C}^{T}=\begin{pmatrix}33&37&32\\60&45&27\\18&38&13\end{pmatrix}$$

By (11.16)

$$\mathbf{A}^{-1}=-\frac{1}{147}\begin{pmatrix}33&37&32\\60&45&27\\18&38&13\end{pmatrix}$$

Substituting into (*) gives

$$\begin{pmatrix}i_1\\i_2\\i_3\end{pmatrix}=-\frac{1}{147}\begin{pmatrix}33&37&32\\60&45&27\\18&38&13\end{pmatrix}\begin{pmatrix}7.5\\-17.5\\16\end{pmatrix}$$

$$=-\frac{1}{147}\begin{pmatrix}(33\times 7.5)+(37\times(-17.5))+(32\times 16)\\(60\times 7.5)+(45\times(-17.5))+(27\times 16)\\(18\times 7.5)+(38\times(-17.5))+(13\times 16)\end{pmatrix}$$

$$=-\frac{1}{147}\begin{pmatrix}112\\94.5\\-322\end{pmatrix}=\begin{pmatrix}-0.762\\-0.643\\2.190\end{pmatrix}$$

Hence $i_1=-0.76A$, $i_2=-0.64A$ and $i_3=2.19A$ (2 d.p.)

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7. (a) Since there are 2 zeros in the second row it is easier to expand along this row, thus:

$$\det\begin{pmatrix}2&3&5\\0&0&6\\1&5&3\end{pmatrix}=-0\left[\det\begin{pmatrix}3&5\\5&3\end{pmatrix}\right]+0\left[\det\begin{pmatrix}2&5\\1&3\end{pmatrix}\right]-6\left[\det\begin{pmatrix}2&3\\1&5\end{pmatrix}\right]$$

$$=0+0-6\left[(2\times 5)-(1\times 3)\right]=-42$$

(b) Similarly since there is a zero along the third row, expand along this row.

$$\det\begin{pmatrix}6&7&1\\1&3&2\\0&1&5\end{pmatrix}=0\left[\det\begin{pmatrix}7&1\\3&2\end{pmatrix}\right]-1\left[\det\begin{pmatrix}6&1\\1&2\end{pmatrix}\right]+5\left[\det\begin{pmatrix}6&7\\1&3\end{pmatrix}\right]$$

$$=0-1\left[(6\times 2)-1\right]+5\left[(6\times 3)-(1\times 7)\right]=44$$

(11.16) $$\mathbf{A}^{-1}=\frac{1}{\det\mathbf{A}}adj\mathbf{A}$$

(c) Expand along the first column since it contains 2 zeros:

$$\det\begin{pmatrix}1&5&1\\0&3&7\\0&2&9\end{pmatrix}=1\left[\det\begin{pmatrix}3&7\\2&9\end{pmatrix}\right]-0\left[\det\begin{pmatrix}5&1\\2&9\end{pmatrix}\right]+0\left[\det\begin{pmatrix}5&1\\3&7\end{pmatrix}\right]$$

$$=1\left[(3\times 9)-(2\times 7)\right]-0+0=13$$

(d) Expanding along the second column

$$\det\begin{pmatrix}9&5&1\\13&0&2\\11&0&3\end{pmatrix}=-5\det\begin{pmatrix}13&2\\11&3\end{pmatrix}+0\det\begin{pmatrix}9&1\\11&3\end{pmatrix}-0\det\begin{pmatrix}9&1\\13&2\end{pmatrix}$$

$$=-5\left[(13\times 3)-(11\times 2)\right]=-85$$

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8. The following shows the MAPLE output but you could use a graphical calculator.

```
> A:=<<1 | 2 | 3 | 4> , <5 | 6 | 7 | 8> , <9 | 10 | 11 | 12> , <13 | 14 | 15 | 16>>;
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$$A := \begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\13&14&15&16\end{bmatrix}$$

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> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected

> det(A);
```

$$0$$

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> B:=matrix([[-1.1, 4.23, 2.67, 7.45,9.62], [19.61, 6.40, 3.12, 11.89,2.36], [-17.5, -9.73, 5.23, 8.54,2.51], [6.19, 2.91, 17.64, 8.93,8.98],[3.98,11.84,4.78,9.85,3.22]]);
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$$B := \begin{bmatrix}-1.1&4.23&2.67&7.45&9.62\\19.61&6.40&3.12&11.89&2.36\\-17.5&-9.73&5.23&8.54&2.51\\6.19&2.91&17.64&8.93&8.98\\3.98&11.84&4.78&9.85&3.22\end{bmatrix}$$

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> det(B);
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$$-509092.3880$$
