Complete solutions of Exercise 11(d)

1. (a) In matrix form we have

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

The augmented matrix can be written

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$$

We execute $R_2 - 2R_1$ to get 0 in place of 2 in row 2. Similarly $R_3 - 3R_1$ to achieve a 0 in place of 3 in row 3:

$$\begin{matrix} R_1 \\ R_2^* = R_2 - 2R_1 \\ R_3^* = R_3 - 3R_1 \end{matrix}\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & -4 & 10 & -14 \end{array}\right)$$

$$\begin{matrix} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - \frac{4}{5}R_2^* \end{matrix}\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ 0 & 0 & 6 & -18 \end{array}\right)$$

From the last row, R_3^{**}, we have

$$6z = -18 \text{ which gives } \quad z = -3$$

From R_2^* we have

$$-5y + 5z = 5$$

$$-5y + \left[5\times(-3)\right] = 5 \text{ gives } y = -4$$

From first row, R_1, we have

$$x + 2y - 3z = 3$$

Substituting $y = -4$ and $z = -3$ gives

$$x + \left[2\times(-4)\right] - \left[3\times(-3)\right] = 3 \text{ gives } x = 2$$

Thus $x = 2,\ y = -4,\ z = -3$. Similarly

(b) $x = 1,\ y = 2,\ z = 3$ (c) $x = 1/2,\ y = 1/4,\ z = 1/8$

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2. The augmented matrix is

$$\left(\begin{array}{cc|c} 9\times 10^3 & -3\times 10^3 & 10 \\ -3\times 10^3 & 13\times 10^3 & 0 \end{array}\right)$$

Divide each row by 10^3 gives

$$\begin{matrix} R_1 \\ R_2 \end{matrix}\left(\begin{array}{cc|c} 9 & -3 & 10\times 10^{-3} \\ -3 & 13 & 0 \end{array}\right)$$

Interchanging row 1 and row 2

$$\begin{matrix} R_1' \\ R_2' \end{matrix}\left(\begin{array}{cc|c} -3 & 13 & 0 \\ 9 & -3 & 10\times 10^{-3} \end{array}\right)$$

To get 0 in place of 9 we execute $R_2' + 3R_1'$

$$\begin{matrix} R_1' \\ R_2'' \end{matrix}\left(\begin{array}{cc|c} -3 & 13 & 0 \\ 0 & 36 & 10\times 10^{-3} \end{array}\right)$$

From R_2'' we have

$$36i_2 = 10\times 10^{-3}$$

$$i_2 = 0.278\times 10^{-3}$$

From R_1' we have

$$-3i_1 + 13i_2 = 0$$

$$i_1 = \frac{13}{3}i_2 = \frac{13}{3}\times 0.278\times 10^{-3} = 1.204\times 10^{-3}$$

Writing to 2 d.p. and using milli (m) gives $i_1 = 1.20mA$ and $i_2 = 0.28mA$.

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3. Writing out the augmented matrix we have

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 10 & 1 & -5 & 18 \\ -20 & 3 & 20 & 14 \\ 5 & 3 & 5 & 9 \end{array}\right)$$

Interchanging row 1 and row 3 gives

$$\begin{matrix} R_1' \\ R_2 \\ R_3' \end{matrix}\left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ -20 & 3 & 20 & 14 \\ 10 & 1 & -5 & 18 \end{array}\right)$$

Executing $R_2 + 4R_1'$ and $R_3' - 2R_1'$ to achieve 0's in place of -20 and 10 respectively:

$$\begin{matrix} R_1' \\ R_2' \\ R_3'' \end{matrix}\left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ 0 & 15 & 40 & 50 \\ 0 & -5 & -15 & 0 \end{array}\right)$$

Only left to replace the -5 in the bottom row by 0, implement $3R_3'' + R_2'$:

$$\begin{matrix} R_1' \\ R_2' \\ R_3''' \end{matrix}\left(\begin{array}{ccc|c} 5 & 3 & 5 & 9 \\ 0 & 15 & 40 & 50 \\ 0 & 0 & -5 & 50 \end{array}\right)$$

From the last row we have $-5z = 50$ gives $z = -10$

Substituting this into R_2' and solving for y gives $y = 30$. Similarly from the first row we have $x = -6.2$.

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4. Rearranging the given equations

$$\begin{aligned} \ddot{x}_1 + \ddot{x}_2 \qquad &= 0 \\ -\ddot{x}_1 \qquad + T &= g \\ -2\ddot{x}_2 + T &= 2g \end{aligned}$$

Putting into an augmented matrix

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & g \\ 0 & -2 & 1 & 2g \end{array}\right)$$

Establishing a 0 in place of -1:

$$\begin{matrix} R_1 \\ R_2' = R_2 + R_1 \\ R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & g \\ 0 & -2 & 1 & 2g \end{array}\right)$$

Getting 0 in place of -2

$$\begin{matrix} R_1 \\ R_2' \\ R_3' = R_3 + 2R_2' \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & g \\ 0 & 0 & 3 & 4g \end{array}\right)$$

From the last row we have

$$3T = 4g \text{ gives } T = \frac{4}{3}g$$

From R_2' and substituting the above we have

$$\ddot{x}_2 + \frac{4}{3}g = g \text{ which gives } \ddot{x}_2 = -\frac{4}{3}g + g = -\frac{g}{3}$$

From the first row, R_1, we have $\ddot{x}_1 = \dfrac{g}{3}$.

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5. The augmented matrix is

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 3g \\ 0 & -2 & 1 & 2g \end{array}\right)$$

$$\begin{matrix} R_1 \\ R_2' = R_2 + 3R_1 \\ R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3g \\ 0 & -2 & 1 & 2g \end{array}\right)$$

$$\begin{matrix} R_1 \\ R_2' \\ R_3' = R_3 + \frac{2}{3}R_2' \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3g \\ 0 & 0 & 5/3 & 4g \end{array}\right)$$

By R_3' we have

$$\frac{5}{3}T = 4g \text{ which gives } T = \frac{12}{5}g$$

Similarly from R_2' and R_1 we have

$$\ddot{x}_1 = -\frac{g}{5},\ \ddot{x}_2 = \frac{g}{5}$$

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6. The augmented matrix is

$$\left(\begin{array}{ccc|c} 4\times10^3 & -3\times10^3 & 0 & 10 \\ -3\times10^3 & 18\times10^3 & -10\times10^3 & 0 \\ 0 & -10\times10^3 & 23\times10^3 & -15 \end{array}\right)$$

Dividing each row by 10^3 gives

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \left(\begin{array}{ccc|c} 4 & -3 & 0 & 10\times10^{-3} \\ -3 & 18 & -10 & 0 \\ 0 & -10 & 23 & -15\times10^{-3} \end{array}\right) \end{array}$$

Interchanging columns I_1 and I_3:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} I_3 & I_2 & I_1 \end{array} \\ \left(\begin{array}{ccc|c} 0 & -3 & 4 & 10\times10^{-3} \\ -10 & 18 & -3 & 0 \\ 23 & -10 & 0 & -15\times10^{-3} \end{array}\right) \end{array}$$

Divide row 2, R_2, by -10 gives

$$\begin{array}{c} R_1 \\ R_2' \\ R_3 \end{array} \left(\begin{array}{ccc|c} 0 & -3 & 4 & 10\times10^{-3} \\ 1 & -1.8 & 0.3 & 0 \\ 23 & -10 & 0 & -15\times10^{-3} \end{array}\right)$$

To get 0 in place of 23 we need to execute $R_3 - 23R_2'$

$$\begin{array}{c} R_1 \\ R_2' \\ R_3' \end{array} \left(\begin{array}{ccc|c} 0 & -3 & 4 & 10\times10^{-3} \\ 1 & -1.8 & 0.3 & 0 \\ 0 & 31.4 & -6.9 & -15\times10^{-3} \end{array}\right)$$

To achieve 0 in place of 31.4 we have to implement $R_3' + \dfrac{31.4}{3} R_1$

$$\begin{array}{c} R_1 \\ R_2' \\ R_3' \end{array} \left(\begin{array}{ccc|c} 0 & -3 & 4 & 10\times10^{-3} \\ 1 & -1.8 & 0.3 & 0 \\ 0 & 0 & 34.967 & 89.667\times10^{-3} \end{array}\right)$$

Interchanging R_1 and R_2' gives 0's in the required position

$$\begin{array}{c} \begin{array}{ccc} I_3 & I_2 & I_1 \end{array} \\ \left(\begin{array}{ccc|c} 1 & -1.8 & 0.3 & 0 \\ 0 & -3 & 4 & 10\times10^{-3} \\ 0 & 0 & 34.967 & 89.667\times10^{-3} \end{array}\right) \end{array}$$

From the last row we have

$$34.967 I_1 = 89.667\times10^{-3} \text{ which gives } I_1 = \frac{89.667\times10^{-3}}{34.967} = 2.564332\times10^{-3}$$

Substituting $I_1 = 2.564332\times10^{-3}$ into the penultimate row gives

$$-3I_2 + \left(4\times2.564332\times10^{-3}\right) = 10\times10^{-3}$$

$$I_2 = \frac{\left(10\times10^{-3}\right) - \left(4\times2.564332\times10^{-3}\right)}{-3} = 8.5776\times10^{-5}$$

From the first row we have

$$I_3 - 1.8I_2 + 0.3I_1 = 0 \qquad (*)$$

Substituting $I_1 = 2.564332\times10^{-3}$ and $I_2 = 8.5776\times10^{-5}$ into (*) yields

$$I_3 - \left(1.8\times 8.5776\times 10^{-5}\right) + \left(0.3\times 2.564332\times 10^{-3}\right) = 0$$

$$I_3 = \left(1.8\times 8.5776\times 10^{-5}\right) - \left(0.3\times 2.564332\times 10^{-3}\right) = -6.149028\times 10^{-4}$$

Thus $I_1 = 2.56\, m\text{A}$, $I_2 = 85.8\,\mu\text{A}$ and $I_3 = -615\,\mu\text{A}$.

Of course the whole solution would look a lot nicer if we had taken advice of the hint and worked in $\text{k}\Omega$ and mA ! No factors of '$\times 10^{-3}$' all over the place.